1.3 Compound Interest: Future Value

YOU WILL NEED

- calculator
- spreadsheet software
- financial application on a graphing calculator

EXPLORE...

- What is the next term in this pattern? How do you know? 100, 150, 225, 337.5, 506.25, ...

compounded annually

When compound interest is determined or paid yearly.

GOAL

Determine the future value of an investment that earns compound interest.

LEARN ABOUT the Math

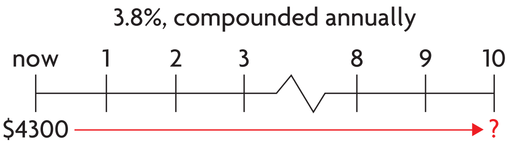
Yvonne earned \$4300 in overtime on a carpentry job. She invested the money in a 10-year Canada Savings Bond that will earn 3.8% **compounded annually**. She decided to invest in a CSB, instead of keeping the money in a savings account, because the CSB will earn more interest.

? What is the future value of Yvonne's investment after 10 years?

EXAMPLE 1 Using reasoning to develop the compound interest formula

Deep's Solution

P is \$4300.
r is 3.8% or 0.038, compounded annually.
t is 0, 1, 2, 3, ..., 10 years.



To organize my thinking, I recorded the information I knew in a timeline. Each space along the timeline represents one year. Each number represents the end of that year.

At the end of year 1:

$A = P(1 + rt)$

$A = 4300(1 + (0.038)(1))$

$A = 4300(1.038)$

$A = 4463.40$

I used the simple interest formula to determine the value of the investment at the end of year 1.

After year 2:

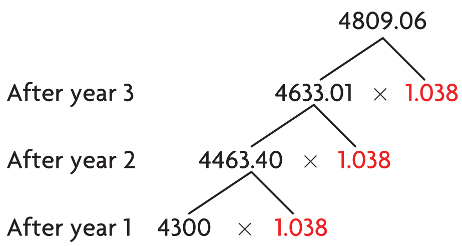
$A = 4463.40(1 + (0.038)(1))$
$A = 4463.40(1.038)$
$A = 4633.01$

I used the simple interest formula again to determine the value at the end of year 2. This time, however, the principal was the value at the end of year 1, \$4463.40.

After year 3:

$A = 4633.01(1 + (0.038)(1))$
$A = 4633.01(1.038)$
$A = 4809.06$

I used the simple interest formula again to determine the value at the end of year 3. This time, the principal was the value at the end of year 2, \$4633.01.



I used a diagram to show the pattern in the future value determinations for each year.

My diagram showed that I had multiplied the principal, \$4300, by a factor of 1.038 three times to determine the value at the end of year 3.

I decided that I could extend the pattern to determine the future value at the end of year 10.

After 3 years:

$A = 4300(1.038)(1.038)(1.038)$
$A = 4300(1.038)^3$
$A = 4809.06$

After 10 years:

$A = 4300(1.038)^{10}$
$A = 6243.699\ldots$

The future value after 10 years is \$6243.70.

I represented the pattern in an equation that shows the future value after 10 years of compounding. The future value is the product of the principal, \$4300, and 1.038 raised to the power of the number of **compounding periods**, 10.

compounding period

The time over which interest is determined; interest can be compounded annually, semi-annually (every 6 months), quarterly (every 3 months), monthly, weekly, or daily.

$6243.699... = 4300(1.038)^{10}$

or

$6243.699... = 4300(1 + 0.038)^{10}$

I realized that 0.038 is the compound interest rate expressed as a decimal.

Let A represent the future value,
P represent the principal,
i represent the interest rate per compounding period,
and n represent the number of compounding periods:
$A = P(1 + i)^n$

I used the pattern to develop a general formula for determining the future value of any investment that earns compound interest.

Reflecting

A. Describe the pattern in the year-by-year calculations of the amount of Yvonne's investment.

B. The compound interest earned (I) on an investment at the end of any compounding period is the difference between the value of the investment at that time (A) and the original principal (P):

$$I = A - P$$

How can this relationship be represented symbolically using the variables I, A, P, i, and n?

C. For Yvonne's investment, the number of compounding periods in the term was the same as the number of years. Suppose that the interest had been compounded semi-annually. How many compounding periods would there have been at maturity? Explain.

APPLY the Math

EXAMPLE 2 Determining the future value of an investment with semi-annual compounding

Matt has invested a $23 000 inheritance in an account that earns 13.6%, compounded semi-annually. The interest rate is fixed for 10 years. Matt plans to use the money for a down payment on a house in 5 to 10 years.

a) What is the future value of the investment after 5 years? What is the future value after 10 years?

b) Compare the principal and the future values at 5 years and 10 years. What do you notice?

c) If the investment had earned simple interest, would the relationship between the principal and the future values have been the same? Explain.

Matt's Solution

a) Annual rate, $r = 13.6\%/\text{a}$

Interest rate over each compounding period,

$$i = \frac{13.6\%}{2}$$

$i = 6.8\%/\text{half year or } 0.068$

Since the interest rate is annual but the compounding period is semi-annual, I determined the semi-annual interest rate by dividing the annual rate by 2.

Term of 5 years:
Number of compounding periods, $n = (5)(2)$
$n = 10$
Principal, P, $= \$23\,000$
Future value after 5 years, $A_5 = P(1 + i)^n$

$$A_5 = 23\,000(1 + 0.068)^{10}$$
$$A_5 = \$44\,405.87$$

Multiplying the term in years by the number of times interest is earned each year gave me the number of compounding periods, n.

I used the compound interest formula to determine the future value of the investment after 5 years and after 10 years.

Term of 10 years:
Number of compounding periods, $n = (10)(2)$
$n = 20$
Future value after 10 years, $A_{10} = P(1 + i)^n$

$$A_{10} = 23\,000(1 + 0.068)^{20}$$
$$A_{10} = \$85\,733.96$$

b) Principal, $P = \$23\,000$
Future value after 5 years, $A_5 = \$44\,405.87$
Future value after 10 years, $A_{10} = \$85\,733.96$

After 5 years, the future value is just less than twice the principal. After 10 years, or double the time, the future value is more than triple the principal.

In the first 5 years, the investment earned \$21 405.87 in interest. In the next 5 years, it earned \$41 328.09 in interest. The difference in the interest earned in the two 5-year periods is due to the compounding of the interest over time.

c) No, the relationship would have been different.
Simple interest:
$I = Prt$

$$I = 23\,000(0.136)(5)$$
$$I = 15\,640$$

With simple interest, \$15 640 would have been earned after 5 years and $2 \cdot \$15\,640$ or \$31 280 would have been earned after 10 years. After 10 years of simple interest, the investment would have earned exactly twice as much interest as it would have earned after 5 years.

Your Turn

Suppose that Matt invested in an account earning 13.6%, compounded quarterly. Predict how the future values at 5 years and 10 years would change. Explain your prediction, and then verify it.

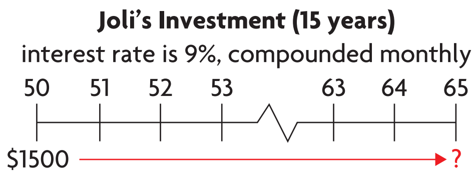
EXAMPLE 3 Determining the future value of investments with monthly compounding

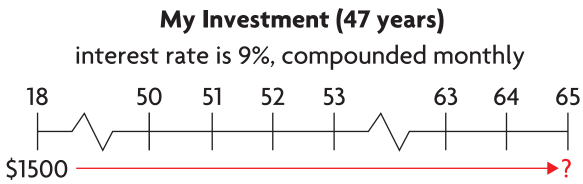
Both Joli, age 50, and her daughter Lena, age 18, plan to invest \$1500 in an account with an annual interest rate of 9%, compounded monthly.

a) If both women hold their investments until age 65, what will be the difference in the future values of their investments?
b) Lena's older step-brother Cody, age 34, also plans to invest \$1500 at 9%, compounded monthly. Determine the future value of his investment at age 65.

Lena's Solution

a)





I drew a timeline for each investment to organize the given information and visualize the problem.

I have about triple the amount of time for my investment to earn interest. I predicted that I will earn quite a bit more than three times the amount of interest, because the interest is compounded.

The annual interest rate is 9% or 0.09.

The monthly interest rate, i, is $\frac{0.09}{12}$ or 0.0075.

Number of compounding periods, n, for Joli's investment:

$(65 - 50)(12) = 180$

Number of compounding periods, n, for my investment:

$(65 - 18)(12) = 564$

Before I could use the future value formula for compound interest, I had to determine the number of compounding periods. To determine the number of compounding periods, I subtracted each age from 65 and then multiplied by 12, since compounding was 12 times a year.

Joli's Investment	My Investment
$A = P(1 + i)^n$ $A = 1500(1 + 0.0075)^{180}$ $A = 5757.064...$	$A = P(1 + i)^n$ $A = 1500(1 + 0.0075)^{564}$ $A = 101\,461.709...$

$101\,461.709... - 5757.064... = 95\,704.644...$

My future value is \$95 704.64 greater.

I used the compound interest formula to determine each future value.

I will earn almost \$100 000 in interest ($101\,461.71 - 1500 \doteq 100\,000$), while my mother will earn only about \$4300 in interest ($5757.06 - 1500 \doteq 4300$).

b) $P = 1500$
$i = 0.0075$
$n = 31 \cdot 12$ or 372

$$A = P(1 + i)^n$$
$$A = 1500(1 + 0.0075)^{372}$$
$$A = 24\ 168.61$$

Cody's investment will have a future value of $24 168.61.

Although Cody's investment will have 31 years to grow (exactly halfway between 15 and 47 years), I predicted that his future value will be a lot less than halfway between the future values of the 15-year and 47-year investments (which is about $53 000) because of compound interest.

This amount seems reasonable, given my prediction.

Your Turn

When Lena invested her money, she knew that she was investing it for a long time. She also knew that banks offer investments at higher interest rates for longer terms, although there are usually more restrictions on when the money can be withdrawn. How much more would Lena earn if she invested $1500 for 47 years at an interest rate of 12%, compounded monthly?

EXAMPLE 4 Comparing interest on investments with different compounding periods

Céline wants to invest $3000 so that she can buy a new car in the next 5 years. Céline has the following investment options:

- **A.** 4.8% compounded annually
- **B.** 4.8% compounded semi-annually
- **C.** 4.8% compounded monthly
- **D.** 4.8% compounded weekly
- **E.** 4.8% compounded daily

Compare the interest earned by each of these options for terms of 1 to 5 years.

Céline's Solution

The principal is $3000. The term is 1 year to 5 years.

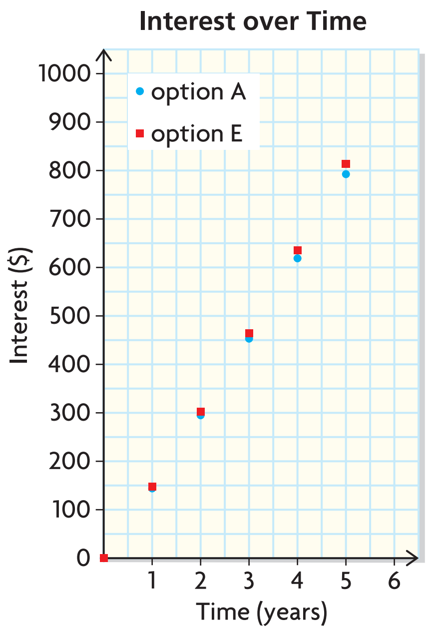
	Compounding Frequency	Compounding Periods per Year
A	annually	1
B	semi-annually	2
C	monthly	12
D	weekly	52
E	daily	365

I predicted that option E would have a greater future value for any length of term because it has the most frequent compounding. The more frequent the compounding, the greater the impact that the compounding will have.

I created a spreadsheet and used the formula $A = P(1 + i)^n$ to determine the future value of each option for 1 to 5 years.

I chose to have the decimal float, so the future values would be as exact as possible.

		A	B	C	D	E
1	**Principal ($)**	3000	3000	3000	3000	3000
2	**Interest Rate per Annum**	0.048	0.048	0.048	0.048	0.048
3	**Periods per Year**	1	2	12	52	365
4	**Value at End of Year**					
5	**0**	3000	3000	3000	3000	3000
6	**1**	3144	3145.728	3147.211	3147.442	3147.502
7	**2**	3294.912	3298.535	3301.645	3302.131	3302.256
8	**3**	3453.068	3458.765	3463.657	3464.422	3464.62
9	**4**	3618.815	3626.777	3633.620	3634.690	3634.966
10	**5**	3792.518	3802.952	3811.922	3813.325	3813.687



I graphed options A and E to compare them visually. I chose options A and E because yearly versus daily compounding would show the greatest difference over time.

I copied the rows of the spreadsheet that showed the future values and then subtracted $3000 (the principal) from each future value.

The graph of each option shows the interest earned increasing in a non-linear way over time. The points for each option grow farther apart over time. Daily compounding (option E) earns the most interest.

It does not matter how many years the investment is held. The more frequent the compounding, the greater the interest earned.

Your Turn

Which representation, the table of values or the graph, do you think better shows the effect of compounding frequency on an investment? Explain.

EXAMPLE 5 Estimating doubling times for investments

Both Berta and Kris invested \$5000 by purchasing Canada Savings Bonds. Berta's CSB earns 8%, compounded annually, while Kris's CSB earns 9%, compounded annually.

a) Estimate the doubling time for each CSB.
b) Verify your estimates by determining the doubling time for each CSB.
c) Estimate the future value of an investment of \$5000 that earns 8%, compounded annually, for 9, 18, and 27 years. How close are your estimates to the actual future values?

Percy's Solution

a) Berta's CSB:

$$\frac{72}{8} = 9$$

It will take about 9 years for Berta's CSB to double in value.

Since both interest rates are compounded annually, I used the **Rule of 72** to estimate.

I divided 72 by the annual interest rate as a percent.

Kris's CSB:

$$\frac{72}{9} = 8$$

It will take about 8 years for Kris's CSBs to double in value.

Rule of 72

A simple formula for estimating the doubling time of an investment; 72 is divided by the annual interest rate as a percent to estimate the doubling time of an investment in years.

The Rule of 72 is most accurate when the interest is compounded annually.

b) Berta's CSB:

The principal is \$5000.
The annual interest rate is 8%.
The compounding frequency is annual, or 1 time per year.
The term (in years) is unknown.
The future value is double \$5000, or \$10 000.

I used the financial application on my calculator and entered these values to determine an exact doubling time for Berta's CSB. I solved for the term, in years.

The doubling time is 9.01 years, which is very close to my estimate of 9 years.

The value is 9.006..., which I rounded to two decimal places.

Kris's CSB:
The principal is \$5000.
The annual interest rate is 9%.
The compounding frequency is annual, or 1 time per year.
The term (in years) is unknown.
The future value is double \$5000, or \$10 000.

The doubling time is 8.043... years, which is very close to the estimate of 8 years.

c) In about $\frac{72}{8}$ or 9 years, the investment doubles, so
- in 9 years, it will be about \$10 000,
- in 18 years, it will be about \$20 000, and
- in 27 years, it will be about \$40 000.

After 9 years, the future value will be \$9995.02.
After 18 years, the future value will be \$19 980.10.
After 27 years, the future value will be \$39 940.31.

I used the financial application on my calculator to determine the exact future values.

The Rule of 72 is less accurate when there is repeated doubling.

Your Turn

Use the Rule of 72 to estimate the doubling time for each investment. Then determine the doubling time. What do you notice about the effect of the compounding frequency on the accuracy of your estimate?

a) \$5000 at 8%, compounded semi-annually
b) \$5000 at 8%, compounded monthly
c) \$5000 at 8%, compounded weekly
d) \$5000 at 8%, compounded daily

In Summary

Key Ideas

- The future value of an investment that earns compound interest can be determined using the compound interest formula

$$A = P(1 + i)^n$$

where A is the future value, P is the principal, i is the interest rate per compounding period (expressed as a decimal), and n is the number of compounding periods.
- The more frequent the compounding and the longer the term, the greater the impact of the compounding on the principal and the greater the future value will be.

Need to Know

- When using the compound interest formula, use an exact value for i. For example, for an annual interest rate of 5% compounded monthly, substitute $\frac{0.05}{12}$ for i instead of the rounded value 0.004 16....
- Four common compounding frequencies are given in the table below. The table shows how the interest rate per compounding period (i) and the number of compounding periods (n) are determined.

Compounding Frequency	Times per Year	Interest Rate per Compounding Period (i)	Number of Compounding Periods (n)
annually	1	i = annual interest rate	n = number of years
semi-annually	2	$i = \frac{\text{annual interest rate}}{2}$	n = (number of years)(2)
quarterly	4	$i = \frac{\text{annual interest rate}}{4}$	n = (number of years)(4)
monthly	12	$i = \frac{\text{annual interest rate}}{12}$	n = (number of years)(12)

- The total compound interest earned on an investment (I) after any compounding period can be determined using the formula

$$I = A - P \quad \text{or} \quad I = P[(1 + i)^n - 1]$$

- The Rule of 72 is a simple strategy for estimating doubling time. It is most accurate when the interest is compounded annually. For example, \$1000 invested at 3% interest, compounded annually, will double in value in about $\frac{72}{3}$ or 24 years; \$1000 invested at 6% will double in about $\frac{72}{6}$ or 12 years.

CHECK Your Understanding

1. Copy and complete the table.

Compound Interest Rate per Annum (%)	Compounding Frequency	Term	Interest Rate per Compounding Period, *i* (%)	Number of Compounding Periods, *n*
10.2	semi-annually	4 years		
4.1	monthly	6 years		
13.2	quarterly	7 years		
3.5	daily	9 months		

2. Determine the future value and the total interest earned for each investment.

a) \$520 invested for 8 years at 4.5% compounded monthly

b) \$1400 invested for 15 years at 8.6% compounded semi-annually

PRACTISING

3. For each investment,

i) use the Rule of 72 to estimate the doubling time and then determine the doubling time.

ii) determine the future value and the total interest earned.

	Principal (*P*) (\$)	Rate of Compound Interest per Annum (%)	Compounding Frequency	Term (years)
a)	7 000	6.8	annually	35
b)	850	9.2	monthly	20
c)	12 500	15.6	weekly	5
d)	40 000	2.7	semi-annually	8

4. When Willa was born, her grandparents set up two investments of \$3000 for her. One earns 9%, compounded annually; the other earns 9%, compounded monthly.

a) Willa is now 18. Determine the current value of each investment.

b) Graph the interest earned over time for both investments on the same grid. Plot at least five points for each investment.

c) How does the compounding frequency affect the growth of interest?

5. Parker wanted to buy a new motorcycle but he had only $6000, half the amount he needed.
   a) Estimate when Parker could buy the motorcycle if he invested his money at 4.8%, compounded annually. Verify your estimate.
   b) Estimate how much sooner he could buy the motorcycle if his investment earned 7.2%, compounded annually. Verify your estimate.

6. Trust funds are investments that are set up for a specific purpose. A local business invested $250 000 in a charitable trust fund so that a school can offer scholarships. The interest rate is 3.8%, compounded semi-annually. Only the interest earned can be used to provide the scholarships. How much is available from the trust fund for scholarships each year?

7. Suppose that you are searching online for the best interest rates on a GIC. You find these rates:
   - Bank A offers 6.6%, compounded annually.
   - Bank B offers 6.55%, compounded semi-annually.
   - Bank C offers 6.5%, compounded quarterly.

   Rank these rates from greatest to least return on an investment of $20 000 for a term of 2 years.

8. Estimate how long it would take for $1000 to grow to $16 000 at each interest rate, compounded annually.
   a) 6%　　b) 12%

9. Angie deposited some money into an account with a fixed rate of interest, compounded annually, for 3 years. The growth of the investment is shown in the table below. What is the annual rate of interest? What was the principal that Angie invested?

End of Year	Value of Investment ($)
1	852.00
2	907.38
3	966.36

10. Solomon bought a $40 000 corporate bond (an investment in the form of a loan to a company that earns interest). The bond earns 4.8%, compounded semi-annually. After 4 years, the interest rate changed to 6%, compounded annually. Determine the value of Solomon's investment after 6 years.

11. On Freda's 16th birthday, she invested $1500 in an account that earns 9%, compounded semi-annually. On her 20th birthday, she moved her investment to an account that paid 11%, compounded monthly. Determine the value of her account on her 22nd birthday.

Term (years)	Rate (%)
1	1.35
2	1.65
3	1.90
4	2.15
5	2.65
6	2.70
7	2.85
8	2.90
9	3.00
10	3.25

12. Lenny has $5000 to invest and is looking at different GICs, as shown in the table to the left. These GICs cannot be redeemed until their maturity.

a) Why do you think the interest rates increase as the term increases?

b) Lenny cannot decide whether to invest $5000 for 10 years or to invest $5000 for 5 years and then reinvest for another 5 years.

i) Compare the future values of each option. What assumptions are you making?

ii) What are the advantages and disadvantages of each option?

Closing

13. Compare simple and compound interest investments by describing what they have in common and what is unique to each.

Extending

14. For each of the past 5 years, Purleen has purchased a $500 Canada Savings Bond on the same date. The interest rate on all the CSBs is 2.9%, compounded semi-annually.

a) What is the current total value of Purleen's CSBs?

b) If Purleen continues this pattern of purchase, what will be the value of her CSBs after another 5 years?

15. This year and on subsequent alternating years, Terry plans to invest $900 in a savings account that earns 11.2%, compounded quarterly. What is the value of his savings immediately after he has made his fourth investment?

History | Connection

Interest Rates by the Decade

Interest rates depend on many economic factors and vary over the years. In Roman times, interest rates typically ranged from 4% to 12% and were paid monthly. It was not rare, however, to have an interest rate that was a multiple of 12, such as 24% or 48%!

Today, the interest rates that are offered by financial institutions, such as banks, are influenced by the economy, both in Canada and the world. On the other hand, interest rates can be used to influence the economy. Interest rates can be lowered to encourage people to borrow and spend in order to stimulate the economy, or they can be raised to encourage saving.

Over the past century, interest rates in Canada have varied from 0% to almost 20%. Canadian banks set their interest rates according to the rate set by the Bank of Canada.

A. Research the Bank of Canada prime interest rates for one decade.

B. Choose the lowest and highest interest rates within the decade. Determine how much more a compound interest investment of $10 000 would earn at the higher rate, compared with the lower rate.

Applying Problem-Solving Strategies

Saving for Retirement

Imagine that you have just started your first full-time job. You have set the financial goal of saving $1 000 000 for your retirement.

YOU WILL NEED

- calculator
- spreadsheet software or financial application on graphing calculator
- standard die

The Game

A. Decide what year you will start acting on the financial goal.

B. Research to determine the current interest rates on savings accounts, GICs, and Canada Savings Bonds. Assume these are the rates that will be available to you when you start acting on your goal.

C. Choose an investment and make an initial deposit of $5000.

D. Determine the future value of your initial investment when the investment matures, or after one year, if the investment has no maturity date.

E. Roll a standard die. If the result is 1 or 2, the current interest rates on all investments decrease by 1%. If the result is 3 or 4, interest rates stay the same. If the result is 5 or 6, interest rates increase by 1%.

F. Based on the new rates, choose an investment and invest whatever you have from the previous investment, plus another $5000 for each year that has passed since your last investment.

G. Repeat steps C through F until you have $1 000 000. How old will you be when you reach your goal?

The Strategy

H. Describe the investment strategy you used to determine the youngest age you will be when you reach your goal.

I. Which factor seems to be more important: the strategy you are using, or the outcome of the die roll? Explain.

Creating a Variation of the Game

J. Can you make the game more realistic? Describe any modifications you could make, and explain the effect you expect from your modifications.

K. Play the game again, using your modified rules. Does the modified game seem to increase or decrease the age at which you reach the goal?