3.5 Conditional Statements and Their Converse

GOAL

Understand and interpret conditional statements.

LEARN ABOUT the Math

James and Gregory like to play soccer, regardless of the weather. Their coach made this **conditional statement** about today's practice:
"If it is raining outside, then we practise indoors."

? When will the coach's conditional statement be true, and when will it be false?

EXPLORE...

Consider the following two statements:

- If Briony is texting, then she is using a cellphone.
- If Briony is using a cellphone, then she is texting.

How do these statements relate to each other? Are they both true?

conditional statement

An "if–then" statement; for example, "If it is Monday, then it is a school day."

EXAMPLE 1 Verifying a conditional statement

Verify when the coach's conditional statement is true or false.

James's Solution: Using reasoning and a truth table

Hypothesis: "It is raining outside."
Conclusion: "We practise indoors."

I identified the **hypothesis** by writing the statement that followed "If."

I identified the **conclusion** by writing the statement that followed "then."

Each of these statements is either true or false, so to verify this conditional statement, I need to consider four cases.

hypothesis

An assumption; for example, in the statement "If it is Monday, then it is a school day," the hypothesis is "It is Monday."

conclusion

The result of a hypothesis; for example, in the statement "If it is Monday, then it is a school day," the conclusion is "it is a school day."

Case 1: The hypothesis is true and the conclusion is true. *It rains outside, and we practise indoors.*
When the hypothesis and conclusion are both true, a conditional statement is true.

The coach kept his promise, so the coach's conditional statement is true.

Case 2: The hypothesis is false, and the conclusion is false. *It does not rain outside, and we practise outdoors.*
When the hypothesis and conclusion are both false, a conditional statement is true.

Since it does not rain outside, the coach is not obligated to keep his promise. We can practise either indoors or outdoors, so the coach's conditional statement is true.

Case 3: The hypothesis is false, and the conclusion is true. *It does not rain outside, and we practise indoors.*
When the hypothesis is false and the conclusion is true, a conditional statement is true.

Since it does not rain outside, the coach is not obligated to keep his promise. We can practise either indoors or outdoors, so the coach's conditional statement is true.

Case 4: The hypothesis is true, and the conclusion is false. *It rains outside, and we practise outdoors.*
When the hypothesis is true and the conclusion is false, a conditional statement is false.

The coach has broken his promise. We should be practising indoors, not outdoors. This **counterexample** shows that the coach's conditional statement is false.

counterexample

An example that disproves a statement; for example, "If it is Monday, then it is a school day" is disproved by the counterexample that there is no school on Thanksgiving Monday. Only one counterexample is needed to disprove a statement.

Let p represent the hypothesis:
It is raining outside.
Let q represent the conclusion:
We practise indoors.

I decided to create a truth table to summarize my observations. I began by representing the hypothesis and conclusion with variables.

p	q	$p \Rightarrow q$
T	T	T
F	F	T
F	T	T
T	F	F

I entered my analysis from each of the four cases above in a truth table.

From the truth table I can see that the only time a conditional statement will be false is when the hypothesis is true and the conclusion is false. This means that when I assume that the hypothesis in a conditional statement is true, I can determine if the conditional statement is true or false based only on whether the conclusion is true or false.

When the hypothesis is false, regardless of whether the conclusion is true or false, the conditional statement is true. When the hypothesis is true and the conclusion is true, the conditional statement is true.

Gregory's Solution: Using reasoning and a Venn diagram

U = {The universal set of all practice times}
P = {The set of all times when it is raining}
Q = {The set of all times when we practise indoors}

I defined the sets I could use in this situation.

Let p represent the hypothesis: *It is raining outside.*
Let q represent the conclusion: *We practise indoors.*

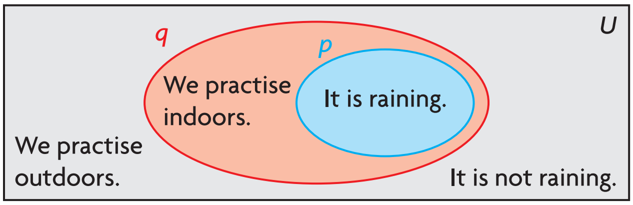
I defined variables to represent the hypothesis and the conclusion.

Suppose t is a practice time; that is, an element of the universal set of all times, U. It follows that:
$P \subset Q$

If $t \in P$, then it is raining at time t, and so $p \Rightarrow q$ means that $t \in Q$. Thus $P \subset Q$.

If $t \notin P$, then it is not raining at time t. In this case, $p \Rightarrow q$ gives no information about whether t is an element of Q.

To represent a conditional statement using a Venn diagram, I drew a small circle for the hypothesis within a large circle for the conclusion.



A Venn diagram supports my reasoning. Clearly the relationship of the two circles is correct. If the conditional statement is true, t is a time when it rains outside, so t belongs inside the small circle. In this case, we practise indoors, so t must also belong inside the larger circle. This can only happen if P is a subset of Q.

Further, if it is not raining outside, then t belongs outside the small circle and may or may not lie inside the large circle. (Both are possible, since we can practise either indoors or outdoors when it is not raining outside.)

From the Venn diagram, I can see that the conditional statement will be true when
- the hypothesis is true, and the conclusion is true.
- the hypothesis is false.

From the Venn diagram, I can see that the only time a conditional statement will be false is when the hypothesis is true and the conclusion is false.

A time t can never lie inside P and outside Q at the same time, which would have to happen if the conditional statement is false.

Reflecting

A. Examine James's truth table.

i) What do you notice about the conditional statement when the hypothesis is false?

ii) What do you notice about the statement when the conclusion is true?

iii) What do you notice about the statement when the hypothesis is true and the conclusion is false?

B. How can you use a Venn diagram to decide if a conditional statement is true or false?

Communication | *Notation*

$p \Rightarrow q$ is notation for "If p, then q."
$p \Rightarrow q$ is read as "p implies q."

APPLY the Math

EXAMPLE 2 | Determining if the converse of a conditional statement is true

Recall the coach's conditional statement: "If it is raining outside, then we will practise indoors."

Is the **converse** of this conditional statement true or false? Justify your decision.

converse
A conditional statement in which the hypothesis and the conclusion are switched; for example, the converse of "If it is Monday, then it is a school day" is "If it is a school day, then it is Monday."

Gregory's Solution

Converse: "If we practise indoors, then it is raining outside."

> I wrote the converse by switching the hypothesis and the conclusion.

We could be practising indoors because the soccer field is under repair, not because it is raining.

> I assumed that the hypothesis "we practise indoors" is true. I needed to decide if the conclusion "it is raining outside" will always be true.

This counterexample proves that the converse of the coach's statement is false.

> I found a counterexample, so the conclusion is false.

Your Turn

Consider the following conditional statement: If a whole number is divisible by 10, then its last digit is 0.

a) Is this conditional statement true or false? Explain.
b) Is the converse of this conditional statement true or false? Explain.

EXAMPLE 3 | Determining if a statement is biconditional

Sayyna told her friend Pipaluk, "If you are north of latitude 60° N, you can experience over 18.8 h of daylight on June 21."

a) Is Sayyna's statement true?
b) Write the converse. Is it true?
c) Is Sayyna's statement **biconditional**?

biconditional
A conditional statement whose converse is also true; in logic notation, a biconditional statement is written as "*p* if and only if *q*." For example, the statement "If a number is even, then it is divisible by 2" is true. The converse, "If a number is divisible by 2, then it is even," is also true. The biconditional statement is "A number is even if and only if it is divisible by 2."

Pipaluk's Solution: Using reasoning

a) The southern borders of Yukon, Northwest Territories, and Nunavut lie along the latitude 60° N.

> I determined the hours of daylight at this latitude on June 21.

On an Hours of Daylight map, the maximum hours of daylight at this latitude is 18 h 53 min.

Let p represent the hypothesis: *You are north of latitude 60° N.*

Let q represent the conclusion: *You can experience over 18.8 h of daylight on June 21.*

p	q	$p \Rightarrow q$
T	T	T

Sayyna's statement "If you are north of latitude 60° N, you can experience over 18.8 h of daylight on June 21" is true.

I assumed that p is true. I need to decide if the conclusion, q, that follows is true.

If you are north of latitude 60° N, you can experience over 18.8 h of daylight on June 21. The conclusion is true.

When the hypothesis is true and the conclusion is true, the conditional statement is true.

b) The converse of Sayyna's statement is "If you can experience over 18.8 h of daylight on June 21, then you are north of latitude 60° N."

To write the converse, I switched the order of the hypothesis and the conclusion.

Based on my original definitions of the variables:

q is now the hypothesis: *You can experience over 18.8 h of daylight on June 21.*

p is now the conclusion: *You are north of latitude 60° N.*

q	p	$q \Rightarrow p$
T	T	T

I assumed that the new hypothesis is true. You experience over 18.8 h of daylight. This means you are north of latitude 60° N. The new conclusion is true.

When the hypothesis is true and the conclusion is true, the conditional statement is true.

The converse of Sayyna' statement, "If you experience over 18.8 h of daylight on June 21, then you are north of latitude 60° N," is also true.

Since Sayyna's conditional statement and its converse are true, the statement is biconditional.

c) $p \Rightarrow q$ is true.

$q \Rightarrow p$ is true.

Therefore, $p \Leftrightarrow q$ is true.

Sayyna's statement is biconditional.

Communication | *Notation*

$p \Leftrightarrow q$ is notation for "p if and only if q."
This means that both the conditional statement and its converse are true statements.

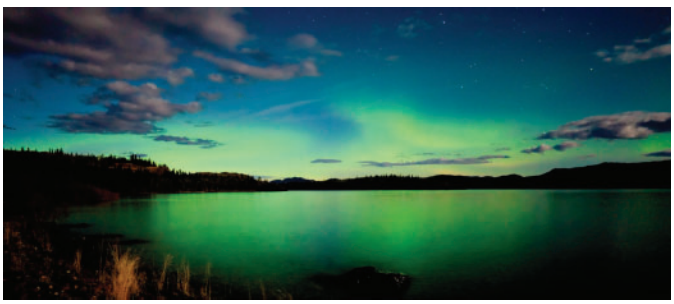
Your Turn

Sanela made this conditional statement: "If you live in Nunavut, you will experience days with less than 5.9 h of daylight."

a) Is Sanela's statement true?

b) Write the converse. Is it true?

c) Is Sanela's statement biconditional?



EXAMPLE 4 Writing conditional statements

"A person who cannot distinguish between certain colours is colour blind."

a) Write this sentence as a conditional statement in "if p, then q" form.
b) Write the converse of your statement.
c) Is your statement biconditional? Explain.

Emile's Solution

a) If p, then q.
If a person cannot distinguish between certain colours, then that person is colour blind.

I wrote the first part of the sentence as p, and the second part as q.

b) If q, then p.
If a person is colour blind, then that person cannot distinguish between certain colours.

I wrote the converse by switching the hypothesis and the conclusion.

c) The first statement is true.
The converse is also true.

I looked up the definition of colour blindness in the dictionary.

Since both the conditional statement and its converse are true, the statement is biconditional.
I can write the statement as:
"A person is colour blind if and only if that person cannot distinguish between certain colours."

A person who cannot distinguish between certain colours is colour blind, and a person who is colour blind cannot distinguish between certain colours.

Your Turn

A square has four right angles.
a) Write this sentence as a conditional statement in "if p, then q" form.
b) Write the converse of your statement.
c) Is your statement biconditional? Explain your reasoning.

EXAMPLE 5 Verifying a biconditional statement

Reid stated the following biconditional statement: "A quadrilateral is a square if and only if all of its sides are equal." Is Reid's biconditional statement true? Explain.

Emanuella's Solution

Conditional statement: "If a quadrilateral is a square, then all of its sides are equal."

I wrote Reid's statement as a conditional statement.

This conditional statement is true.

If I assume that a quadrilateral is a square, then the conclusion that it has four equal sides is also true. Since the conclusion is true when the hypothesis is true, the conditional statement is true.

"If all the sides of a quadrilateral are equal, then it is a square."

I wrote the converse.

A counterexample is a rhombus. Therefore, the converse is false.

If I assume that all the sides of a quadrilateral are equal, then the conclusion that the shape is a square is false. A rhombus has four equal sides, but a rhombus is only a square if all four angles measure 90°. Since the conclusion is false when the hypothesis is true, the conditional statement is false.

Since the converse of the conditional statement is false, the biconditional statement is false.

Your Turn

Meredith wrote the following biconditional statement: "A quadrilateral is a parallelogram if and only if its opposite sides are parallel." Is Meredith's biconditional statement true? Explain.

EXAMPLE 6 Making a decision based on a conditional financial statement

Brian and Anna want to buy a house. They have determined that they can afford to make a mortgage payment of $1400 each month.

a) If the current interest rate is 4%, the term of the mortgage is 5 years, and the mortgage will be amortized over 25 years, then what is the maximum mortgage they can afford?

b) If the interest rate doubled in 5 years, then how would this increase affect the maximum mortgage they might consider today? What advice can you give?

Bill's Solution: Using technology

a) The number of payments is 25 · 12 or 300.
The interest rate is 4%.
The present value is unknown.
The payments are $1400.
The future value is $0.
The payment frequency is 12.
The compounding frequency is 2.

I used the finance application on my calculator to determine the maximum mortgage payment.

I knew that in Canada, all mortgage interest is compounded semi-annually.

Therefore, the present value is $266 149.47. The maximum mortgage they should consider is $266 149.47.

b) The number of payments is 25 · 12 or 300.
The interest rate is 8%.
The present value is unknown.
The payment is $1400.
The future value is $0.
The payment frequency is 12.
The compounding frequency is 2.

Therefore, the present value is $183 434.92.

I determined the present value for a 25-year mortgage at 8%. Since they will have paid off some of the principal of their mortgage over the first 5 years at 4%, they will actually be able to carry a mortgage a little greater than $183 434.92 if interest rates increase to 8% for the next 20 years. However, to be safe, they should not get a mortgage for any more than $185 000.

If the interest rate doubled, the mortgage might be too costly. I determined the maximum mortgage that they could afford today while keeping their monthly payment at $1400. I used 8% as the interest rate over the entire 25-year period.

To ensure that they can still afford the monthly payments on the house in 5 years, when the mortgage comes up for renewal, they should get a smaller mortgage. They should look for a less expensive house.

Your Turn

a) Generate your own "what if" question, dealing with the financial considerations of Anna and Brian purchasing a house. Make a decision regarding your question. Justify your decision.
b) Do you think that Bill's advice is logically justified? Explain.

In Summary

Key Ideas

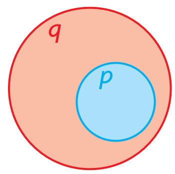
- A conditional statement consists of a hypothesis, p, and a conclusion, q. Different ways to write a conditional statement include the following:
  - If p, then q.
  - p implies q.
  - $p \Rightarrow q$
- To write the converse of a conditional statement, switch the hypothesis and the conclusion.

Need to Know

- A conditional statement is either true or false. A truth table for a conditional statement, $p \Rightarrow q$, can be set up as follows:

p	q	$p \Rightarrow q$
T	T	T
F	F	T
F	T	T
T	F	F

  A conditional statement is false only when the hypothesis is true and the conclusion is false. Otherwise, the conditional statement is true, even if the hypothesis is false.
- You can represent a conditional statement using a Venn diagram, with the inner oval representing the hypothesis and the outer oval representing the conclusion. The statement "p implies q" means that p is a subset of q.



- Only one counterexample is needed to show that a conditional statement is false.
- If a conditional statement and its converse are both true, you can combine them to create a biconditional statement using the phrase "if and only if."

CHECK *Your Understanding*

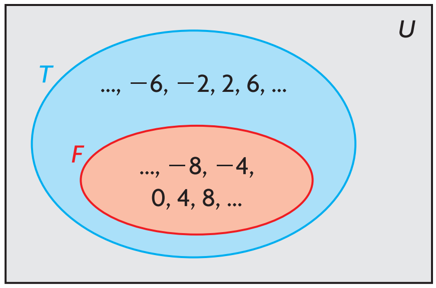
1. Consider the following conditional statement: "If I am swimming in the ocean, then I am swimming in salt water."

a) Write the hypothesis and the conclusion.

b) Is the conditional statement true? If it is false, provide a counterexample.

c) Write the converse. Is the converse true? If it is false, provide a counterexample.

2. Use the Venn diagram to answer the following questions.



   a) Consider the following conditional statement in relation to the Venn diagram: "If a number is divisible by 4, then it is divisible by 2." Is this statement true?
   b) Write the converse. Is the converse true?
   c) Determine a counterexample, if possible.

3. An equilateral triangle has three equal sides.
   a) Write this statement in "if p, then q" form.
   b) Write the converse of your conditional statement in part a).
   c) Is each statement true or false?
   d) Is the statement biconditional? Explain.

PRACTISING

4. A Spanish proverb says, "Since we cannot get what we like, let us like what we can get."
   a) Write the proverb in "if p, then q" form.
   b) What is the hypothesis? What is the conclusion?

5. Consider this conditional statement: "If a number is divisible by 5, then its final digit is a 0."
   a) Is this statement true?
   b) Write the converse.
   c) Is the converse true? Support your decision with a Venn diagram.

If you are in the capital of Canada, then you can visit Parliament.

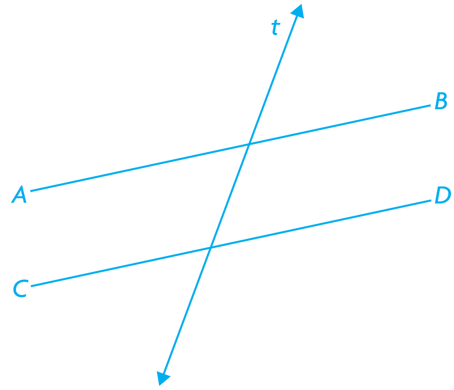
6. Determine whether each statement is biconditional. If the statement is biconditional, rewrite it in biconditional form. If the statement is not biconditional, provide a counterexample.
   a) If you live in Canada, then you live in North America.
   b) If you live in the capital of Canada, then you live in Ottawa.

7. Use a truth table to determine whether the following statement is biconditional: If $\sqrt{x^2} = x$, then x is not negative.

8. Write each statement in "if p, then q" form. If the statement is biconditional, rewrite it in biconditional form. If the statement is not biconditional, provide a counterexample.
   a) A half-empty glass is half full.
   b) A rhombus has equal opposite angles.
   c) A repeating decimal can be expressed as a fraction.

9. A transversal, t, intersects line segments AB and CD. Consider this statement:
"The line segments AB and CD are parallel if and only if the alternate angles are equal."

a) Write a conditional statement and its converse.
b) Are the statements you wrote in part a) true or false? Explain how you know.
c) Is the original statement true or false? Explain how you know.



10. Write the converse of each statement. Then determine if each statement is biconditional:

a) If your pet barks, then it is a dog.
b) If your pet is a dog, then it wags its tail.

11. Is each statement true or false?

a) If $x + y = z$, then $x = z - y$.
b) If $p - q = r$, then $q + r = p$.

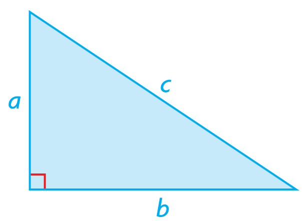
12. Consider the Sudoku puzzle. Write a conditional statement about the number that belongs in the shaded square. What conclusion can you make about where to place the numbers in that column?

	4			2	7			
	7		3	6	5		2	
2			9				7	
6		9						3
	1						9	
5	2					7		6
	3				4			8
	9		7	5	6		3	
			2	8			6	

13. For each statement below,

i) write the statement in "if p, then q" form,
ii) write the converse of the statement,
iii) verify the statement and its converse, and
iv) if the statement and its converse are both true, write the biconditional statement.

a) A square has four right angles.
b) For a right triangle, $a^2 + b^2 = c^2$.



c) A trapezoid has two sides that are parallel.

14. **a)** Michelle and Marc are buying a house. If they require a $250 000 mortgage, amortized for 25 years, at 6.5% compounded semi-annually, then:

i) Determine the amount of each mortgage payment if they make one payment a month.

ii) Determine the amount of each mortgage payment if they make semi-monthly payments.

b) If Michelle and Marc make payments twice a week, but pay the same total amount per month they would need to pay for semi-monthly payments, then how many mortgage payments would they make? What advice would you give Michelle and Marc?

Closing

15. **a)** Write two different statements in "if p, then q" form. One of these statements should be biconditional.

b) Represent each statement with a Venn diagram.

c) Explain how to tell whether the converse of each statement is true or false using the Venn diagram.

Extending

16. A cryptoquote is a code-breaking puzzle. In a cryptoquote, every word must contain a vowel and no letter ever represents itself. For example, consider the cryptoquote below:

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AC   NG   CL   FCA   AC   NG
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This cryptoquote is broken as follows:

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To   be   or   not   to   be
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To solve a cryptoquote, look for repeated letters, especially in two-letter words. A word that contains only one letter must be either "a" or "I."

a) Write an "if-then" statement for the two-letter words in a cryptoquote.

b) Solve the cryptoquote below. For this cryptoquote, assume that "J" represents "a."

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KSQ QSCAXHBMV TD TY DKJD CSNSAU
CXXA QJTD J YTCLVX PSPXCD NXBSHX
YDJHDTCL DS TPZHSWX DKX QSHVA.
     – JCCX BHJCG
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17. Suppose that Brian and Anna, from Example 6, decide to get a mortgage for the maximum amount they are allowed: $266 149.47. If they decide to pay an additional $250 per month to reduce the time required to pay off the mortgage, and the interest rate stays at 4%, then:

a) How many months will it take them to pay off the mortgage?

b) How much money will they save in total?