3.6 The Inverse and the Contrapositive of Conditional Statements

EXPLORE...

Mother Teresa said, "If you judge people, you have no time to love them."

- Decide whether this statement means the same as the following statement: "If you don't judge people, you have time to love them."

GOAL

Understand and interpret the contrapositive and inverse of a conditional statement.

INVESTIGATE the Math

Puneet's math teacher said, "If a polygon is a triangle, then it has three sides."

Puneet's geography teacher said, "If you live in Saskatoon, then you live in Saskatchewan."

She wonders what other statements she can write using this information.

? What other variations can Puneet write, and are they true?

A. Begin with the math teacher's statement. Is it true? Explain.

B. Write the converse of the math teacher's statement. Is it true? Explain.

C. Write the **inverse** of the math teacher's statement. Is it true? Explain.

D. Write the **contrapositive** of the math teacher's statement. Is it true? Explain.

E. Repeat parts A to D using the geography teacher's statement.

inverse

A statement that is formed by negating both the hypothesis and the conclusion of a conditional statement; for example, for the statement "If a number is even, then it is divisible by 2," the inverse is "If a number is **not** even, then it is **not** divisible by 2."

contrapositive

A statement that is formed by negating both the hypothesis and the conclusion of the **converse** of a conditional statement; for example, for the statement "If a number is even, then it is divisible by 2," the contrapositive is "If a number is **not** divisible by 2, then it is **not** even."

Reflecting

F. If you are given a conditional statement that you know is true, can you predict whether

i) the converse is true?

ii) the inverse is true?

iii) the contrapositive is true?

G. Test your conjectures from part F using this true statement: "If a quadrilateral is a rectangle, then it is a parallelogram."

H. Examine the statements you wrote for the inverse and converse for each of the conditional statements given by Puneet's teachers. What do you notice?

APPLY the Math

EXAMPLE 1 Verifying the inverse and contrapositive of a conditional statement

Consider the following conditional statement: "If today is February 29, then this year is a leap year."

a) Verify the statement, or disprove it with a counterexample.
b) Verify the inverse, or disprove it with a counterexample.
c) Verify the contrapositive, or disprove it with a counterexample.

Maggie's Solution: Using a truth table

a) "If today is February 29, then this year is a leap year."

Hypothesis (p): *Today is February 29.*
Conclusion (q): *This year is a leap year.*
Conditional statement: If p, then q.

I examined the conditional statement.

I determined the hypothesis, p, and the conclusion, q.

If I assume that the hypothesis, p, is true, then the conditional statement is true only when the conclusion, q, is also true.

p	q	$p \Rightarrow q$
T	T	T

Assuming that today is February 29 means that the hypothesis is true. Therefore, the conclusion is also true since February has 28 days, unless it is a leap year. Then February has 29 days.

This conditional statement is true.

b) Inverse: "If today is not February 29, then this year is not a leap year."

To write the inverse, I negated both the hypothesis and the conclusion.

Hypothesis ($\neg p$): *Today is not February 29.*
Conclusion ($\neg q$): *This year is a not a leap year.*
If $\neg p$, then $\neg q$.

$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$
T	F	F

Assuming that it is not February 29 means that the hypothesis is true. Therefore, the conclusion is false. For example, today could be January 5, 2012, but 2012 is a leap year. This counterexample shows that the inverse is false.

The inverse is false.

Communication | *Notation*

In logic notation, the inverse of "if p, then q" is written as "If $\neg p$, then $\neg q$."

c) Contrapositive: "If this year is not a leap year, then today is not February 29."

Hypothesis ($\neg q$): *This year is a not a leap year.*
Conclusion ($\neg p$): *Today is not February 29.*
If $\neg q$, then $\neg p$.

To write the contrapositive, I switched the hypothesis and the conclusion in the inverse.

Communication | *Notation*

In logic notation, the contrapositive of "if p, then q" is written as "If $\neg q$, then $\neg p$."

$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T

The contrapositive is true.

Assuming that this year is not a leap year means that the hypothesis is true. Therefore, the conclusion is true since February 29 does not occur in any year but a leap year.

Your Turn

Consider the following conditional statement: "If this year is a leap year, then February has 29 days." Determine if the statement, its inverse, and its contrapositive are true or false.

EXAMPLE 2 Examining the relationship between a conditional statement and its contrapositive

Consider the following conditional statement: "If a number is a multiple of 10, then it is a multiple of 5."

a) Write the contrapositive of this statement.
b) Verify that the conditional and contrapositive statements are both true.

Bailey's Solution: Using reasoning

a) Conditional statement: "If a number is a multiple of 10, then it is a multiple of 5."

Contrapositive: "If a number is not a multiple of 5, then it is not a multiple of 10."

I wrote the contrapositive by negating both parts of the converse.
This is the same as negating the hypothesis and the conclusion, then switching their positions in the statement.

b) If n is a multiple of 10, then $n = 10m, m \in I$.

If I assume that a number is a multiple of 10, then the conclusion that the number is a multiple of 5 is true.

Since 10 is a factor of $10m$, n is divisible by 10.
Also, $n = 5(2m), m \in I$.
Since 5 is a factor of $5(2m)$, 5 is a factor of n.
The conditional statement is true.

Contrapositive: "If a number is not a multiple of 5, then it is not a multiple of 10."

The contrapositive is true.

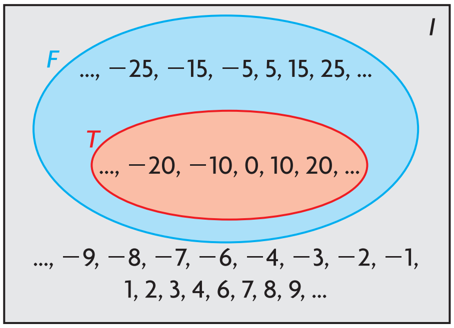
The conditional and the contrapositive statements are both true.

> Next, I considered the contrapositive.
>
> If I assume that a number is not a multiple of 5, then the conclusion that it is not a multiple of 10 is true.
>
> Since all multiples of 10 are also multiples of 5, a number that is not a multiple of 5 cannot be a multiple of 10. The contrapositive is true.

Briony's Solution: Using a Venn diagram

a) $I = \{n \in \mathrm{I}\}$
$F = \{5n, n \in \mathrm{I}\}$
$T = \{10n, n \in \mathrm{I}\}$

> I defined the universal set I of integers, set F, the multiples of 5, and set T, the multiples of 10.



> I drew a Venn diagram to show how sets I, F, and T are related.

Conditional statement: "If a number is a multiple of 10, then it is a multiple of 5."

Contrapositive: "If a number is not a multiple of 5, then it is not a multiple of 10."

b) $T \subset F$
Therefore, the conditional statement is true.

> I examined my Venn diagram.
>
> The conditional statement shows that any number in subset T is in set F.

Any number that is not a multiple of 5 lies in F'.
$T \not\subset F'$
Therefore, the contrapositive is also true.

> My Venn diagram shows that if a number is in F', it cannot be also in T. Therefore, F' and T are disjoint sets.

The conditional and contrapositive statements are both true.

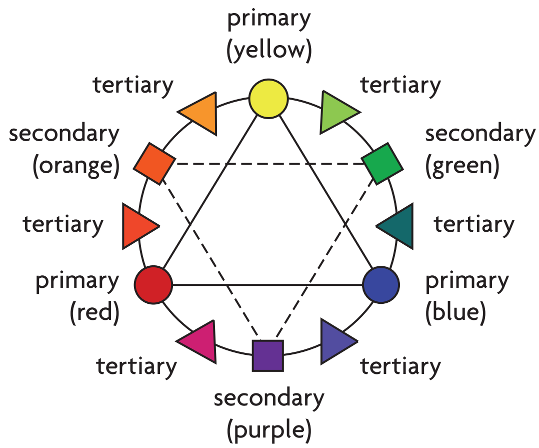
> I can represent both statements with the same Venn diagram.

Your Turn

You are given a conditional statement that you know is true. What can you conclude about the contrapositive?

EXAMPLE 3 Examining the relationship between the converse and inverse of a conditional statement

Arizona is studying the colour wheel in art class. She observes the following: "If a colour is red, yellow, or blue, then it is a primary colour."



a) Write the converse of this statement.
b) Write the inverse of this statement.
c) Verify that the converse and the inverse are both true.
d) Is Arizona's statement biconditional? Explain.

John's Solution: Analyzing statements

a) Conditional statement: "If a colour is red, yellow, or blue, then it is a primary colour."

Converse: "If a colour is a primary colour, then it is red, yellow, or blue."

I wrote the converse by switching the hypothesis and the conclusion in Arizona's statement.

b) Inverse: "If a colour is not red, yellow, or blue, then it is not a primary colour."

I wrote the inverse by negating both the hypothesis and the conclusion in Arizona's statement.

c) The converse is true.

I assumed that a colour was a primary colour, so the hypothesis is true. The only primary colours are red, yellow, and blue, so the conclusion is true. When the hypothesis and conclusion are both true, the conditional statement is true.

The inverse is true.
The converse and inverse are both true.

I assumed that a colour was not red, yellow, or blue, so the hypothesis is true. The only primary colours are red, yellow, and blue. If a colour is not one of these three colours, it is not a primary colour, so the conclusion is true. When the hypothesis and conclusion are both true, the conditional statement is true.

d) A colour is red, yellow, or blue if and only if it is primary. Therefore, the statement is biconditional.

Since red, yellow, and blue are the only primary colours, the statement is biconditional.

This makes sense, since the original statement and its converse are both true.

Arizona's Solution: Using a Venn diagram

a) Converse: "If a colour is a primary colour, then it is red, yellow, or blue."

b) Inverse: "If a colour is not red, yellow, or blue, then it is not a primary colour."

C = {all colours on the colour wheel}

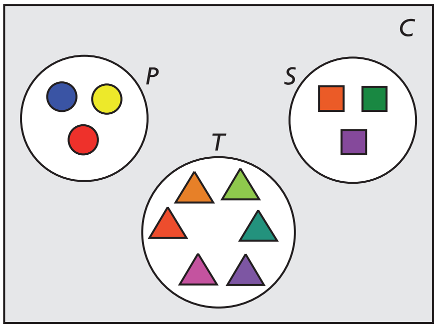
P = {red, yellow, blue}

S = {green, orange, purple}

T = {red-orange, yellow-orange, yellow-green, blue-green, blue-violet, red-violet}

> To write the converse, I switched the hypothesis and the conclusion.
>
> To write the inverse, I negated them.

> I defined the universal set C, then grouped and defined the colours on the wheel in three sets.



> I drew a Venn diagram to show how the sets are related. In this case, the three sets of colours on the colour wheel are disjoint sets.

c) Red, yellow, and blue are in the primary colour circle. There are no other colours in this circle.
Therefore, the converse is true.

All other colours, which are not primary colours, are either in set S or set T.
Therefore, the inverse is true.

The converse and the inverse are both true.

> I examined my Venn diagram.
>
> The converse shows that only red, yellow, and blue are primary colours.

> The inverse shows that if a colour is not red, yellow, or blue, it is not a primary colour.

> I can represent both statements with the same Venn diagram.

d) A colour is red, yellow, or blue if and only if it is primary. Therefore, the statement is biconditional.

> Since red, yellow, and blue are the only primary colours, the statement is biconditional.

Your Turn

You are given the converse of a conditional statement that you know is true. What can you conclude about the inverse?

In Summary

Key Ideas

- You form the inverse of a conditional statement by negating the hypothesis and the conclusion.
- You form the contrapositive of a conditional statement by exchanging and negating the hypothesis and the conclusion.

Need to Know

- If a conditional statement is true, then its contrapositive is true, and vice versa.
- If the inverse of a conditional statement is true, then the converse of the statement is also true, and vice versa.

CHECK Your Understanding

1. Write the converse, inverse, and contrapositive of each conditional statement.

a) If you find success before work, then you are looking in a dictionary.

b) If you are over 16, then you can drive.

c) If a quadrilateral is a square, then its diagonals are perpendicular.

d) If n is a natural number, then $2n$ is an even number.

2. Consider the following conditional statement: "If an animal has a long neck, then it is a giraffe."

a) Write the converse and the contrapositive of this statement.

b) Are the conditional and contrapositive statements both true? Explain.

3. Consider this statement: "If a polygon has five sides, then it is a pentagon."

a) Write the converse and the inverse.

b) Are the converse and the inverse both true? Explain.

4. Jeb claims that this statement is true: If $x^2 = 25$, then $x = 5$.

a) Do you agree or disagree with Jeb? Explain.

b) Is the converse true? Explain.

c) Is the inverse true? Explain.

d) Is the contrapositive true? Explain.

PRACTISING

5. For each conditional statement below,

i) determine if it is true,
ii) write the converse and determine if it is true,
iii) write the inverse and determine if it is true, and
iv) write the contrapositive and determine if it is true.

If any statement is false, provide a counterexample.

a) If you are in Hay River, then you are in the Northwest Territories.
b) If a puppy is male, then it is not female.
c) If the Edmonton Eskimos won every game this season, then they would be number 1 in the West.
d) If an integer is not negative, then it is positive.

6. Complete the following table for the statements in question 5 by indicating whether each statement is true or false.

	Conditional Statement	Inverse	Converse	Contrapositive
a)				
b)				
c)				
d)				

7. Examine your table for question 6.

a) What do you notice about each conditional statement and its contrapositive?
b) What do you notice about the inverse and the converse?

8. Examine your table for question 6 again.

a) What conclusion can you draw about each conditional statement and its converse?
b) What conclusion can you draw about the inverse and the contrapositive?

9. Consider this statement: If a polygon is a square, then the polygon is a quadrilateral.

a) Write the converse, the inverse, and the contrapositive.
b) Verify that each statement is true, or disprove it with a counterexample.

10. Consider this statement: "If the equation of a line is $y = 5x + 2$, then its y-intercept is 2."

a) Write the converse, the inverse, and the contrapositive.
b) Verify that each statement is true, or disprove it with a counterexample.

11. Suppose that a conditional statement, its inverse, its converse, and its contrapositive are all true. What do you know about the conditional statement?

12. For each conditional statement below,

i) verify it, or disprove it with a counterexample,
ii) verify the converse, or disprove it with a counterexample,
iii) verify the inverse, or disprove it with a counterexample, and
iv) verify the contrapositive, or disprove it with a counterexample.

a) If the Moon is a balloon, then a pin can burst the Moon.
b) If x is a negative number, then $-x$ is a positive number.
c) If a number is a perfect square, then it is positive.
d) If a number can be expressed as a terminating decimal, then it can be expressed as a fraction.
e) If the equation of a function is $f(x) = 5x^2 + 10x + 3$, then its graph is a parabola.
f) If a number is an integer, then it is a whole number.
g) If I am 18, then I am old enough to vote.
h) If I am a Canadian, then I enjoy hockey.

Closing

13. Explain, in your own words, why each statement is true.

a) When a conditional statement is true, its contrapositive will be true.
b) When the converse of a conditional statement is true, its inverse will be true.

Extending

14. a) Write a false conditional statement. Show, by counterexample, that the contrapositive is also false.
b) Write a true conditional statement. Show that the contrapositive is also true.

15. a) Write a conditional statement whose inverse is false. Show, by counterexample, that the converse is also false.
b) Write a conditional statement whose inverse is true. Show that the converse is also true.