5.5 Conditional Probability

YOU WILL NEED

- calculator

EXPLORE...

- Jackie plays on a volleyball team called the Giants. The Giants are in a round-robin tournament with five other teams. The teams that they will play against will be selected at random. Determine the probability that their first game will be against the Clippers and their second game will be against the Maroons.

GOAL

Understand and solve problems that involve dependent events.

LEARN ABOUT the Math

A computer manufacturer knows that, in a box of 100 chips, 3 will be defective. Jocelyn will draw 2 chips, at random, from a box of 100 chips.

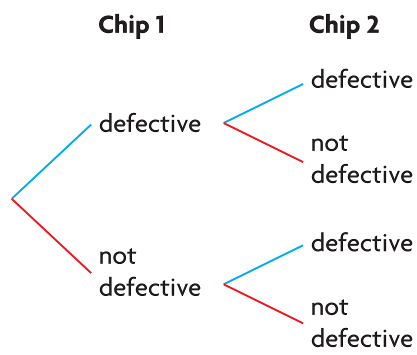
? What is the probability that both of the chips will be defective?

EXAMPLE 1 Calculating the probability of two events

Determine the probability that Jocelyn will draw 2 defective chips.

Jocelyn's Solution

I drew a tree diagram to represent the ways that I could draw 2 computer chips from the box.



There are 4 permutations: both are defective; the first is defective and the second is not defective; the first is not defective and the second is defective; and both are not defective. I am concerned only with the situation where both are defective.

Let A represent the event that the first chip I draw will be defective. Let B represent the event that the second chip I draw will be defective.

I defined the two **dependent events** in this situation.

There are 3 defective chips in the box of 100 chips.

$$P(A) = \frac{3}{100}$$

I determined the probability that the first chip I draw will be defective.

If the first chip I do draw is defective, then the box now holds 99 chips, 2 of which are defective.

$$P(B \mid A) = \frac{2}{99}$$

I determined the **conditional probability** that the second chip I draw will be defective.

dependent events

Events whose outcomes are affected by each other; for example, if two cards are drawn from a deck without replacement, the outcome of the second event depends on the outcome of the first event (the first card drawn).

conditional probability

The probability of an event occurring given that another event has already occurred.

Communication *Tip*

$P(B \mid A)$ is the notation for a conditional probability. It is read "the probability that event B will occur, given that event A has already occurred."

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

$$P(A \text{ and } B) = \left(\frac{3}{100}\right)\left(\frac{2}{99}\right)$$

$$P(A \text{ and } B) = \frac{6}{9900} \text{ or } \frac{1}{1650}$$

There is one chance in 1650 that both of the chips I draw will be defective.

I needed to determine the probability that I will draw 2 defective chips.

I knew that you determine the probability of two independent events occurring by multiplying their probabilities. I reasoned that I could determine the probability of two dependent events occurring in the same way. The difference is that one of the probabilities will be conditional if the events are dependent.

Verify:

$$P(A \text{ and } B) = \frac{3 \cdot 2}{100 \cdot 99}$$

$$P(A \text{ and } B) = \frac{6}{9900} \text{ or } \frac{1}{1650}$$

I got the same answer.

I verified my conjecture using the Fundamental Counting Principle. The probability that I will randomly select 2 defective chips, without replacement, from the box is the number of ways of drawing the 2 defective chips ($3 \cdot 2$) divided by the number of ways of drawing any 2 chips ($100 \cdot 99$).

Reflecting

A. In this example, how does the probability that the first event will occur affect the probability that the second event will occur? Explain.

B. Suppose that Jocelyn replaced the first chip before drawing the second chip. Would the probability of the second chip being defective remain the same? Explain.

C. When determining the probability of drawing two defective chips, Jocelyn did not consider the possibility that the first chip she drew would not be defective. Explain why.

D. Explain why the probability of drawing a defective chip on the second draw is considered a conditional probability.

E. Starting with a tree diagram like Jocelyn's, label each branch with its probability. Determine the probability of drawing each permutation of defective and not defective chips, then add these probabilities. What does the sum imply?

APPLY the Math

EXAMPLE 2 Calculating the conditional probability of a pair of dependent events

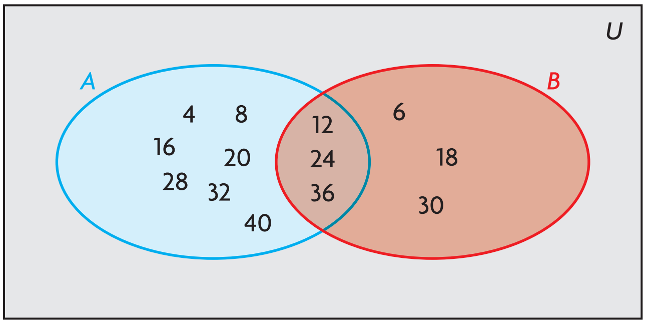
Nathan asks Riel to choose a number between 1 and 40 and then say one fact about the number. Riel says that the number he chose is a multiple of 4. Determine the probability that the number is also a multiple of 6, using each method below.

a) A Venn diagram **b)** A formula

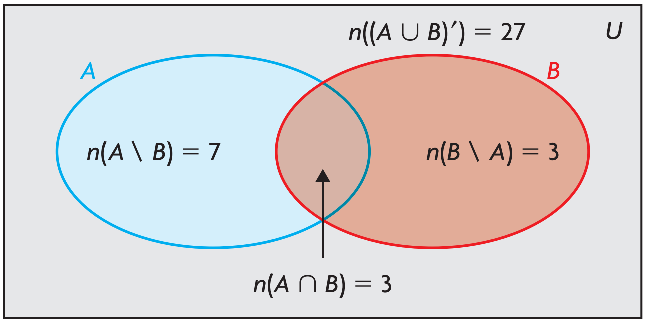
Nathan's Solution

a) Let U = {all natural numbers from 1 to 40}.
Let A = {multiples of 4 from 1 to 40}.
Let B = {multiples of 6 from 1 to 40}.

I defined the universal set and the events.



I drew a Venn diagram to show the sets. I wrote all the numbers that Riel could have chosen in my diagram.



I redrew the Venn diagram to show the number of elements in each area.

The universal set has 40 elements, and $A \cup B$ has 13 elements.

Therefore, $(A \cup B)'$ must have $40 - 13$ or 27 elements.

Riel could have chosen one of 10 numbers. Only three of these numbers are multiples of 4 and 6.

Since the probability that Riel's number is a multiple of 6 is $\frac{3}{10}$,

$$P(B \mid A) = \frac{3}{10}$$

I knew that 10 of the numbers from 1 to 40 are multiples of 4. This is $n(A)$, which is the total number of outcomes for the conditional probability.

Three of these numbers are also multiples of 6. This is $n(A \cap B)$, which is the number of favourable outcomes for the conditional probability.

b) $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

I substituted $(A \cap B)$ into the formula.

$P(A \cap B) = P(A) \cdot P(B \mid A)$

$\frac{P(A \cap B)}{P(A)} = P(B \mid A)$

I determined the values of $P(A)$ and $P(A \cap B)$ using my Venn diagram.

$P(A) = \frac{10}{40}$

$P(A \cap B) = \frac{3}{40}$

$\frac{\frac{3}{40}}{\frac{10}{40}} = P(B \mid A)$

$\frac{3}{10} = P(B \mid A)$

This is the same answer that I got from the Venn diagram.

I wrote the formula for determining the probability of two dependent events.

From my Venn diagram, I saw that the area of overlap, or the intersection, is (A and B).

I solved for $P(B \mid A)$ to develop a formula for determining a conditional probability.

I substituted these values into my formula to determine $P(B \mid A)$.

This makes sense because there are 3 numbers in $(A \cap B)$ and 10 numbers in set A. Since $\frac{3}{40}$ and $\frac{10}{40}$ have the same denominator, I got $\frac{3}{10}$ when I divided these probabilities.

Your Turn

Nathan chose another number from 1 to 40 and told Riel that it is a multiple of 6. Determine the probability that this number is also a multiple of 4.

EXAMPLE 3 Solving a conditional probability problem

According to a survey, 91% of Canadians own a cellphone. Of these people, 42% have a smartphone. Determine, to the nearest percent, the probability that any Canadian you met during the month in which the survey was conducted would have a smartphone.

Tara's Solution

I let C represent owning a cellphone.
I let S represent owning a smartphone.

I defined the events.

$P(C) = 0.91$

$P(S \mid C) = 0.42$

I wrote the probability that a Canadian owns a cellphone and the conditional probability that a Canadian owns a smartphone, given that she or he owns a cellphone.

$$P(S \mid C) = \frac{P(S \cap C)}{P(C)}$$

$$0.42 = \frac{P(S \cap C)}{0.91}$$

$$(0.42)(0.91) = P(S \cap C)$$

$$0.3822 = P(S \cap C)$$

I substituted the known probabilities into the formula for conditional probability.

I solved for $P(S \cap C)$.

Smartphones are a subset of cellphones, so it makes sense that $P(S)$ is equal to $P(S \cap C)$.

$$P(S) = P(S \cap C)$$

$$P(S) = 38.22\%$$

To the nearest percent, the probability that any Canadian I met in that month would have a smartphone is 38%.

Your Turn

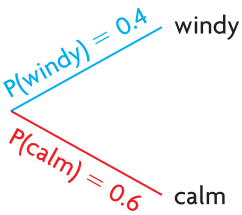
a) Determine, to the nearest percent, the probability that any Canadian you met in that month would have a cellphone but not a smartphone.
b) How could you represent this probability in a Venn diagram?

EXAMPLE 4 Making predictions that involve dependent events

Hillary is the coach of a junior ultimate team. Based on the team's record, it has a 60% chance of winning on calm days and a 70% chance of winning on windy days. Tomorrow, there is a 40% chance of high winds. There are no ties in ultimate. What is the probability that Hillary's team will win tomorrow?

Hillary's Solution

P(windy) is 40%, so P(calm) is 100% − 40% or 60%.

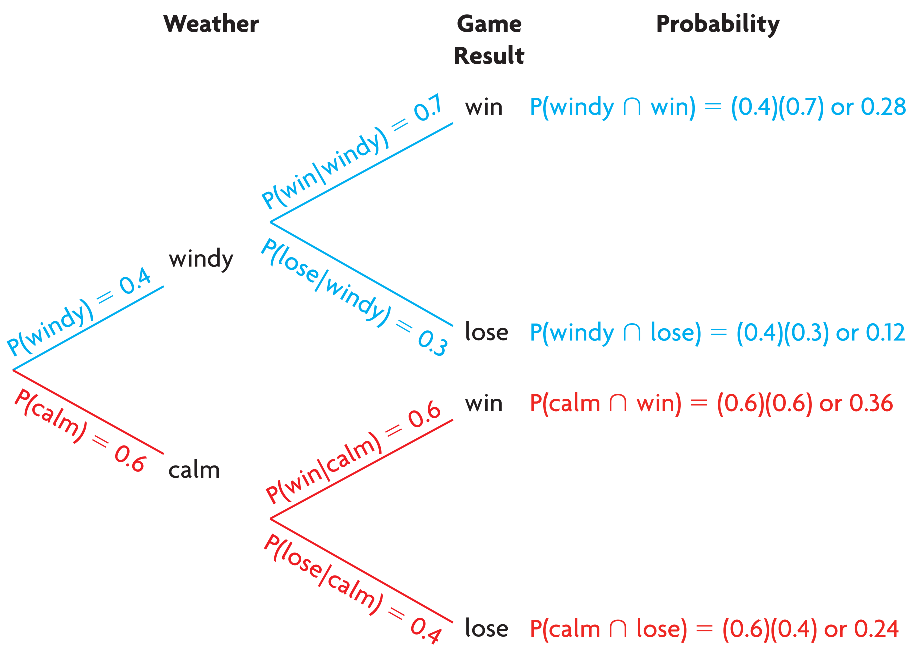


I determined the probability of a calm day tomorrow.

Then I drew the first branches of my tree diagram.

$P(\text{win} \mid \text{windy}) = 70\%$
$P(\text{lose} \mid \text{windy}) = 100\% - 70\%$ or 30%
$P(\text{win} \mid \text{calm}) = 60\%$
$P(\text{lose} \mid \text{calm}) = 100\% - 60\%$ or 40%

I determined the probabilities of losing on windy days and calm days. Since there are no ties in ultimate, these events are complementary.



I completed my tree diagram by multiplying the probabilities along each branch. I knew that I could do this because the events on each branch are pairs of dependent events.

$P(\text{win}) = P(\text{windy} \cap \text{win}) + P(\text{calm} \cap \text{win})$
$P(\text{win}) = 0.28 + 0.36$
$P(\text{win}) = 0.64$

The probability that we will win tomorrow is 64%.

To determine the probability that we will win tomorrow, I added both probabilities that would result in a win. I could do this because the two events (windy and win, and calm and win) are mutually exclusive. Only one of these events will occur.

Your Turn

a) Explain how you can check quickly to determine if Hillary made any errors in her tree diagram.

b) Determine the probability that Hillary's team will lose tomorrow. Verify your solution using another method.

In Summary

Key Ideas

- If the probability of one event depends on the probability of another event, then these events are called **dependent events**. For example, drawing a heart from a standard deck of 52 playing cards and then drawing another heart from the same deck without replacing the first card are dependent events.
- If event B depends on event A occurring, then the **conditional probability** that event B will occur, given that event A has occurred, can be represented as follows:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Need to Know

- If event B depends on event A occurring, then the probability that both events will occur can be represented as follows:

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

- A tree diagram is often useful for modelling problems that involve dependent events.
- Drawing an item and then drawing another item, without replacing the first item, results in a pair of dependent events.

CHECK Your Understanding

1. Austin rolls a regular six-sided red die and a regular six-sided black die. If the red die shows 4 and the sum of the two dice is greater than 7, Austin wins a point.

a) Are the two events dependent or independent?

b) Determine the probability that Austin will win a point.

2. Valeria draws a card from a well-shuffled standard deck of 52 playing cards. Then she draws another card from the deck without replacing the first card.

a) Are these two events dependent or independent?

b) Determine the probability that both cards are diamonds.

3. Valeria draws a card from a well-shuffled standard deck of 52 playing cards. Then she puts the card back in the deck, shuffles again, and draws another card from the deck.

a) Are these two events dependent or independent?

b) Determine the probability that both cards are diamonds.

PRACTISING

4. Lexie has six identical black socks and eight identical white socks loose in her drawer. She pulls out one sock at random and then another sock, without replacing the first sock.

a) Determine the probability of each event below.

i) She pulls out a pair of black socks.

ii) She pulls out a pair of white socks.

iii) She pulls out a matched pair of socks; that is, either both are black or both are white.

b) If Lexie randomly pulled out both socks at the same time, would your answers for part a) change? Explain.

5. There are 80 males and 110 females in the graduating class in a Kelowna school. Of these students, 30 males and 50 females plan to attend the University of British Columbia (UBC) next year.

a) Determine the probability that a randomly selected student plans to attend UBC.

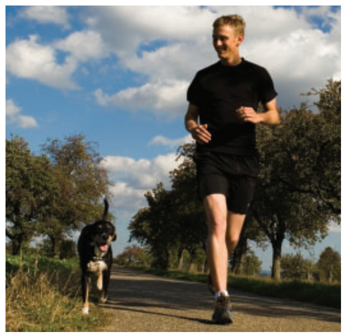
b) A randomly selected student plans to attend UBC. Determine the probability that the selected student is female.

6. Each day, Melissa's math teacher gives the class a warm-up question. It is a true-false question 30% of the time and a multiple-choice question 70% of the time. Melissa gets 60% of the true-false questions correct, and 80% of the multiple-choice questions correct. Melissa answers today's question correctly. What is the probability that it was a multiple-choice question?

7. Skye has four loonies, three toonies, and five quarters in his pocket. He needs two loonies for a parking meter. He reaches into his pocket and pulls out two coins at random. Determine the probability that both coins are loonies.

8. Anita remembers to set her alarm clock 62% of the time. When she does remember to set her alarm clock, the probability that she will be late for school is 0.20. When she does not remember to set it, the probability that she will be late for school is 0.70. Anita was late today. What is the probability that she remembered to set her alarm clock?

9. Ian likes to go for daily jogs with his dog, Oliver. If the weather is nice, he is 85% likely to jog for 8 km. If the weather is rainy, he is only 40% likely to jog for 8 km. The weather forecast for tomorrow indicates a 30% chance of rain. Determine the probability that Ian will jog for 8 km.



10. Cellphone users in Mapletown were surveyed about their phone plans, with these results:
- 70% of all users have call display.
- 40% of users have a data plan.
- 75% users with a data plan also have call display.

A cellphone user in Mapletown, who is selected at random, has call display. Determine the probability that this person also has a data plan.

11. Cole surveyed 10 students in his Grade 12 class about their lunch breaks on school days. He asked them to base their answers to the following questions on a period of 1 month.
- How often do you have 1 h or less for lunch?
- How often do you have more than 1 h for lunch?
- How often do you go to your local fast-food outlet for lunch, if you only have 1 h or less for lunch?
- How often do you go to your local fast-food outlet for lunch, if you have more than 1 h for lunch?

His results are given to the right:

A = {1 h or less for lunch}

B = {more than 1 h for lunch}

C = {go to fast food outlet if less than 1 h for lunch}

D = {go to fast food outlet if more than 1 h for lunch}

Event	Total Number of Lunches
A	120
B	80
C	40
D	60

Create a conditional probability problem for Cole's data.

12. Decide on a topic that interests you.

a) Create and conduct a short survey, similar to Cole's survey in the previous question. Tabulate your results.

b) Create a conditional probability problem for your data.

13. The probability that a car tire will last for 5 years is 0.8. The probability that a tire will last for 6 years is 0.5. Suppose that your parents' tires have lasted for 5 years. Determine the probability that the tires will last for 6 years.

14. The probability that the windshield wipers on a particular model of car will last for 2 years is 0.7. The probability that they will last for 3 years is 0.6. The wipers on your parents' car have lasted for 2 years. Determine the probability that the wipers will last for 3 years.

15. The probability that a particular pair of badminton shoes will last for 6 months is 0.9. The probability that the shoes will last for 1 year is 0.2. Natalie's shoes have lasted for 6 months. Determine the probability that they will last for 1 year.

16. Morgan asks Jasmine to choose a number from 20 to 50 and then say one fact about the number. Jasmine says that the number she chose is a multiple of 5. Determine the probability that the number is also a multiple of 3, using each method below.

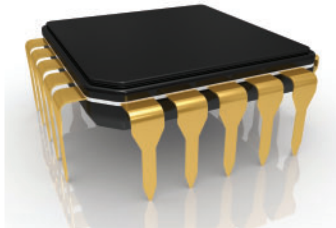
a) A Venn diagram

b) The formula for conditional probability

17. Recall the opening problem in this lesson:
A computer manufacturer knows that, in a box of 100 chips, 3 will be defective. Jocelyn will draw 2 chips, at random, from a box of 100 chips.

a) Draw a tree diagram to represent this situation.

b) Determine the probability that exactly 1 chip will be defective. Explain what you did.



18. A computer manufacturer knows that, in a box of 150 computer chips, 3 will be defective. Samuel will draw 2 chips, at random, from a box of 150 chips. Determine the probability that Samuel will draw the following:

a) 2 defective chips

b) 2 non-defective chips

c) Exactly 1 defective chip

19. Savannah's soccer team is playing a game tomorrow. Based on the team's record, it has a 50% chance of winning on rainy days and a 60% chance of winning on sunny days. Tomorrow, there is a 30% chance of rain. Savannah's soccer league does not allow ties.

a) Determine the probability that Savannah's team will win tomorrow.

b) Determine the probability that her team will lose tomorrow.

20. Think of two situations in your life in which the probability of one event happening depends on another event happening. Write two problems, one for each of these situations. Also, write the solutions to your problems. Exchange your problems with a classmate. Solve, and then correct, each other's problems. Adjust your problems, if necessary.

Closing

21. Explain the meaning of the formula $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$. Give an example to illustrate your explanation.

Extending

22. A computer manufacturer knows that, in a box of 100 computer chips, 4 will be defective. Caleb will draw 3 chips, at random, from a box of 100 chips. Determine the probability that Caleb will draw the following:

a) 3 defective chips

b) 3 non-defective chips

c) More defective chips than non-defective chips