5.6 Independent Events

YOU WILL NEED

- calculator

EXPLORE...

- The Fortin family has two children. Cam determines the probability that the family has two girls. Rushanna determines the probability that the family has two girls, given that the first child is a girl. How are these probabilities similar, and how are they different?

GOAL

Understand and solve problems that involve independent events.

INVESTIGATE the Math

Anne and Abby each have 19 marbles: 11 red and 8 blue. Anne places 7 red marbles and 3 blue marbles in bag 1. She places the rest of her marbles in bag 2. Abby places all of her marbles in bag 3. Anne then draws one marble from bag 1 and one marble from bag 2. Abby draws two marbles from bag 3, without replacement.

? Are both girls equally likely to draw two red marbles?

A. Let A represent Anne drawing a red marble from bag 1.
Let B represent Anne drawing a red marble from bag 2.
Are A and B independent or dependent events? Explain.

B. Let C represent Abby drawing a red marble from bag 3 on her first draw. Let D represent Abby drawing a red marble from bag 3 on her second draw. Are C and D independent or dependent events? Explain.

C. Determine $P(A)$ and $P(B)$.

D. Determine $P(A \cap B)$ to three decimal places.

E. Determine $P(C)$ and $P(D \mid C)$, the probability that Abby will draw a red marble from bag 3 on the second draw, given that the first marble she drew was red.

F. Determine $P(C \cap D)$ to three decimal places.

G. Are both girls equally likely to draw two red marbles? Explain.

Reflecting

H. Explain how you can tell if two events are independent or dependent.

I. Suppose that Abby were to draw both of her marbles together, instead of drawing one after the other. Would this affect the probability of Abby drawing two red marbles? Justify your answer.

J. Naveen used the following formula to determine the probability of Anne drawing two red marbles:

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

Would this formula give the correct answer? Explain.

APPLY the Math

EXAMPLE 1 Determining probabilities of independent events

Mokhtar and Chantelle are playing a die and coin game. Each turn consists of rolling a regular die and tossing a coin. Points are awarded for rolling a 6 on the die and/or tossing heads with the coin:

- 1 point for either outcome
- 3 points for both outcomes
- 0 points for neither outcome

Players alternate turns. The first player who gets 10 points wins.

a) Determine the probability that Mokhtar will get 1, 3, or 0 points on his first turn.

b) Verify your results for part a). Explain what you did.

Chantelle's Solution

a) Let S represent rolling a 6 on the die.
Let H represent tossing heads with the coin.

$P(S) = \frac{1}{6}$ $\quad$ $P(H) = \frac{1}{2}$

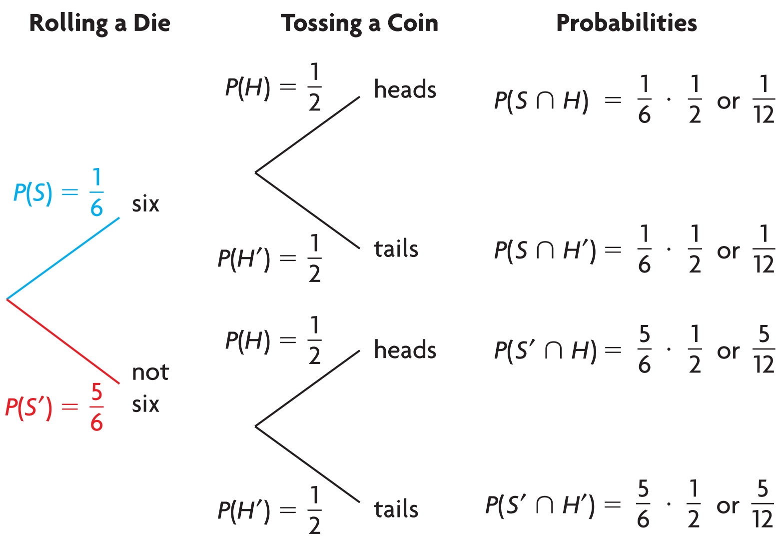
I defined events S and H. These events are independent. The probability of tossing heads with the coin is not affected by the probability of rolling 6 on the die.

$P(S') = 1 - \frac{1}{6}$ $\quad$ $P(H') = 1 - \frac{1}{2}$

$P(S') = \frac{5}{6}$ $\quad$ $P(H') = \frac{1}{2}$

I determined the probability that Mokhtar will not get either outcome when rolling the die and tossing the coin. These events are complementary.

I created a tree diagram showing the probabilities associated with the events of rolling a six on a die, S, and tossing heads with a coin, H, along with the probabilities of their complements, S' and H'.



I determined the probability of each pair of events occurring on the tree diagram by multiplying.

P(scores 3 points): $P(S \cap H) = \frac{1}{12}$

Mokhtar gets 3 points when he rolls a 6 and tosses heads.

P(scores 0 points): $P(S' \cap H') = \frac{5}{12}$

He gets 0 points when he does not roll a 6 and does not toss heads.

P(scores 1 point): $P(S' \cap H) \cup P(S \cap H') = \frac{5}{12} + \frac{1}{12}$

$P(S' \cap H) \cup P(S \cap H') = \frac{6}{12}$ or $\frac{1}{2}$

He gets 1 point when he either rolls a 6 or tosses heads

He has a $\frac{1}{12}$ chance of scoring 3 points, a $\frac{1}{2}$ chance of scoring 1 point, and a $\frac{5}{12}$ chance of scoring 0 points.

b) Verify:

Total probability $= \frac{1}{12} + \frac{5}{12} + \frac{6}{12}$

Total probability $= \frac{12}{12}$ or 1

I verified my solution by adding the three probabilities. The sum is 1, so I knew that my answer is correct.

Your Turn

a) Are you more likely to get points or not get points on each turn? Explain.

b) Would the probabilities determined for Mokhtar's first turn change for his next turn? Explain.

EXAMPLE 2 Solving a problem that involves independent events using a graphic organizer

All 1000 tickets for a charity raffle have been sold and placed in a drum. There will be two draws. The first draw will be for the grand prize, and the second draw will be for the consolation prize. After each draw, the winning ticket will be returned to the drum so that it might be drawn again. Max has bought five tickets. Determine the probability, to a tenth of a percent, that he will win at least one prize.

Max's Solution: Using complements

Let X represent winning the grand prize.
Let Y represent winning the consolation prize.
Let Z represent winning at least one prize.

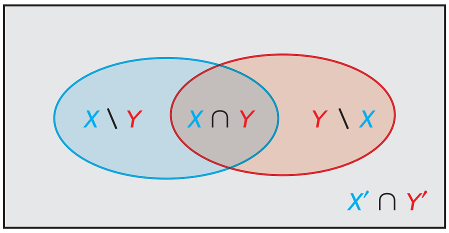
I defined events X, Y, and Z.

$$P(X) = \frac{5}{1000} \text{ or } \frac{1}{200}$$
$$P(Y) = \frac{5}{1000} \text{ or } \frac{1}{200}$$

Since the winning ticket will be replaced after the first draw, the probability of winning either prize is the same.

The events are independent.

In other words, winning the consolation prize is not affected by winning the grand prize.



I drew a Venn diagram to represent the situation.

The event of winning at least one prize, *Z*, is represented by being in any area of the two ovals in the Venn diagram.

$P(Z)$, the probability of winning at least one prize, is the complement of winning no prizes:

$$P(Z) = 1 - P(X' \cap Y')$$

I had to determine the probability of winning the grand prize or the consolation prize.

The only other possibility is winning no prizes.

$$P(X') = 1 - \frac{1}{200} \text{ or } \frac{199}{200}$$
$$P(Y') = 1 - \frac{1}{200} \text{ or } \frac{199}{200}$$
$$P(X' \cap Y') = P(X') \cdot P(Y')$$
$$P(X' \cap Y') = \frac{199}{200} \cdot \frac{199}{200}$$
$$P(X' \cap Y') = \frac{39\,601}{40\,000}$$

I determined the probability of not winning the grand prize, the probability of not winning the consolation prize, and the probability of winning neither prize.

$$P(Z) = 1 - P(X' \cap Y')$$
$$P(Z) = 1 - \frac{39\,601}{40\,000}$$
$$P(Z) = \frac{40\,000}{40\,000} - \frac{39\,601}{40\,000}$$
$$P(Z) = \frac{399}{40\,000} \text{ or } 0.009\,975$$

I determined the probability of winning at least one prize.

The probability I will win at least one prize is about 1.0%.

I rounded to the nearest tenth of a percent.

Melissa's Solution: Using a tree diagram

Let X represent winning the grand prize.
Let Y represent winning the consolation prize.
Let Z represent winning at least one prize.

I defined events X, Y, and Z.

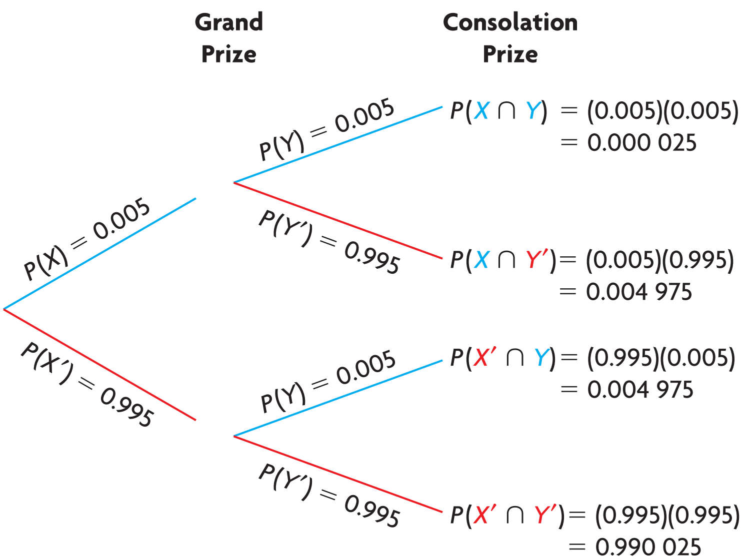
$$P(X) = \frac{5}{1000} \text{ or } 0.005$$

$$P(Y) = \frac{5}{1000} \text{ or } 0.005$$

I determined the probability of winning the grand prize and the probability of winning the consolation prize.

$$P(X') = 1 - P(X) \qquad P(Y') = 1 - P(Y)$$
$$P(X') = 1 - 0.005 \qquad P(Y') = 1 - 0.005$$
$$P(X') = 0.995 \qquad P(Y') = 0.995$$

I also determined the complement of these events: the probability of not winning the grand prize and the probability of not winning the consolation prize.



I drew a tree diagram to list all the possible outcomes. The blue branches represent winning a prize, and the red branches represent not winning a prize. I determined the probability of each pair of events by multiplying the individual probabilities.

Based on my tree diagram, Max could win two prizes in one way.

$$P(\text{win 2 prizes}) = P(X \cap Y)$$
$$P(\text{win 2 prizes}) = 0.000\ 025$$

Max could win one prize in two ways.

$$P(\text{win 1 prize}) = P(X \cap Y') \text{ or } P(X' \cap Y)$$
$$P(\text{win 1 prize}) = P(X \cap Y') \cup P(X' \cap Y)$$
$$P(\text{win 1 prize}) = 0.004\ 975 + 0.004\ 975$$
$$P(\text{win 1 prize}) = 0.009\ 95$$

These probabilities are equal and can be added together, since the two events (only winning the grand prize or only winning the consolation prize) are mutually exclusive.

Winning at least one prize means that Max would win exactly one prize or both prizes.

$P(Z) = P(\text{win 1 prize})$ or $P(\text{win 2 prizes})$
$P(Z) = P(\text{win 1 prize}) \cup P(\text{win 2 prizes})$
$P(Z) = P(\text{win 1 prize}) + P(\text{win 2 prizes})$
$P(Z) = 0.009\ 95 + 0.000\ 025$
$P(Z) = 0.009\ 975$

These probabilities are different and can be added together, since the two events (winning one prize and winning two prizes) are mutually exclusive.

The probability Max will win at least one prize is 1.0%.

I rounded to the nearest tenth of a percent.

Verify:
Total probability $= P(\text{win at least one}) + P(\text{win none})$
Total probability $= 0.009\ 975 + 0.990\ 025$
Total probability $= 1$

I verified my calculations by adding the probabilities. The sum is 1, so I knew that my answer is correct.

Your Turn

Suppose that the rules for the raffle are changed, so the first ticket drawn is not returned to the drum before the second draw.

a) Are the events winning the grand prize and winning the consolation prize dependent or independent?
b) Do you think Max's probability of winning at least one prize will be greater or less than before? Justify your answer.

In Summary

Key Ideas

- If the probability of event *B* does not depend on the probability of event *A* occurring, then these events are called independent events. For example, tossing tails with a coin and drawing the ace of spades from a standard deck of 52 playing cards are independent events.
- The probability that two independent events, *A* and *B*, will both occur is the product of their individual probabilities:

$$P(A \cap B) = P(A) \cdot P(B)$$

Need to Know

- A tree diagram is often useful for modelling problems that involve independent events.
- Drawing an item and then drawing another item, after replacing the first item, results in a pair of independent events.

CHECK Your Understanding

1. For each situation, classify the events as either independent or dependent. Justify your classification.

a) A four-colour spinner is spun, and a die is rolled. The first event is spinning red, and the second event is rolling a 2.

b) A red die and a green die are rolled. The first event is rolling a 1 on the red die, and the second event is rolling a 5 on the green die.

c) Two cards are drawn, without being replaced, from a standard deck of 52 playing cards. The first event is drawing a king, and the second event is drawing an ace.

d) There are 30 cards, numbered 1 to 30, in a box. Two cards are drawn, one at a time, with replacement. The first event is drawing a prime number, and the second event is drawing a number that is a multiple of 5.

2. Celeste goes to the gym five days a week. Each day, she does a cardio workout using either a treadmill, an elliptical walker, or a stationary bike. She follows this with a strength workout using either free weights or the weight machines. Celeste randomly chooses which cardio workout and which strength workout to do each day.

a) Are choosing a cardio workout and choosing a strength workout dependent or independent events? Explain.

b) Determine the probability that Celeste will use a stationary bike and free weights the next day she goes to the gym.

3. Ian also goes to the gym five days a week, but he does two different cardio workouts each day. His choices include using a treadmill, a stepper, or an elliptical walker, and running the track.

a) Are the two cardio workouts that Ian chooses dependent or independent events?

b) Determine the probability that the next time Ian goes to the gym he will use the elliptical walker and then run the track.

PRACTISING

4. For each situation described in question 1, determine the probability that both events will occur.

5. a) Suppose that $P(A) = 0.35$, $P(B) = 0.4$, and $P(A \cap B) = 0.12$. Are A and B independent events? Explain.

b) Suppose that $P(Q) = 0.720$, $P(R) = 0.650$, and $P(Q \cap R) = 0.468$. Are Q and R independent events? Explain.

6. There are two children in the Angel family.
   a) Draw a tree diagram that shows all the possible gender combinations for the two children.
   b) Determine the probability that both children are boys.
   c) Determine the probability that one child is a boy and the other child is a girl.

7. A particular game uses 40 cards from a standard deck of 52 playing cards: the ace to the 10 from the four suits. One card is dealt to each of two players. Determine the probability that the first card dealt is a club and the second card dealt is a heart. Are these events independent or dependent?

8. A standard die is rolled twice. Determine the probability for the following:
   a) The first roll is a 1, and the second roll is a 6.
   b) The first roll is greater than 3, and the second roll is even.
   c) The first roll is greater than 1, and the second roll is less than 6.

9. Jeremiah is going on a cruise up the Nile. According to the travel brochure, the probability that he will see a camel is $\frac{4}{5}$, and the probability that he will see an ibis is $\frac{3}{4}$. Determine the probability that Jeremiah will see the following:
   a) A camel and an ibis
   b) Neither a camel nor an ibis
   c) Only one of these sights

10. a) Design a spinner so that when you toss a coin and spin the spinner, the probability of getting heads and spinning a 6 is $\frac{1}{12}$.
    b) Repeat part a) with a probability of $\frac{1}{20}$.

11. Recall Anne and Abby, from the beginning of this lesson. They each have 19 marbles: 11 red and 8 blue. Anne places 7 red marbles and 3 blue marbles in bag 1. She places the rest of her marbles in bag 2. Abby places all of her marbles in bag 3. Anne then draws one marble from bag 1 and one marble from bag 2. Abby draws two marbles from bag 3.
    a) Are Anne and Abby equally likely to draw two blue marbles from their bags? Explain.
    b) Determine the probability Anne and Abby will both draw one red marble and one blue marble. Explain what you did.
    c) Suppose that Anne now has 5 red marbles and 5 blue marbles in each of her two bags, while Abby has 10 red marbles and 10 blue marbles in her one bag. Will Abby still be more likely to draw two red marbles? Explain.

12. A paper bag contains a mixture of three types of treats: 10 granola bars, 7 fruit bars, and 3 cheese strips. Suppose that you play a game in which a treat is randomly taken from the bag and replaced, and then a second treat is drawn from the bag. You are allowed to keep the second treat only if it was the same type as the treat that was drawn the first time. Determine the probability of each of the following:

a) You will be able to keep a granola bar.
b) You will be able to keep any treat.
c) You will not be able to keep any treat.

13. Tiegan's school is holding a chocolate bar sale. For every case of chocolate bars sold, the seller receives a ticket for a prize draw. Tiegan has sold five cases, so she has five tickets for the draw. At the time of the draw, 100 tickets have been entered. There are two prizes, and the ticket that is drawn for the first prize is returned so it can be drawn for the second prize.

a) Determine the probability that Tiegan will win both prizes.
b) Determine the probability that she will win no prizes.

14. a) Create a problem that involves determining the probability of two independent events. Give your problem to a classmate to solve.
b) Modify the problem you created in part a) so that it now involves two dependent events. Give your problem to a classmate to solve.

15. Two single-digit random numbers (0 to 9 inclusive) are selected independently. Determine the probability that their sum is 10.

Closing

16. a) Explain why the formula you would use to calculate $P(A \cap B)$ would depend on whether A and B are dependent or independent events.
b) Give an example of how you would calculate $P(A \cap B)$ if A and B were independent events.
c) Give an example of how you would calculate $P(A \cap B)$ if A and B were dependent events.

Extending

17. A particular machine has 100 parts. Over a year, the probability that each part of the machine will fail is 1%. If any part fails, the machine will stop.

a) Determine the probability that the machine will operate continuously for 1 year.
b) Suppose that the probability of each part failing within one year dropped to 0.5%. Determine the probability that the machine would operate continuously for 1 year.
c) Suppose that the probability of the machine operating continuously for 1 year must be 90%. What would the probability of not failing need to be for each part?

18. In the final series of the Stanley Cup Playoffs, the first team to win four games becomes the National Hockey League (NHL) champion. The Original Six teams in the NHL were the Boston Bruins, the Chicago Blackhawks, the Detroit Red Wings, the Montréal Canadiens, the New York Rangers, and the Toronto Maple Leafs.

a) In 1960, Montréal played Toronto and won the first four games. Suppose that the probability of Montréal winning any single game was 0.65. Determine the probability that Montréal would win the series in four games.

b) In 1967, the NHL added six more teams, including the St. Louis Blues. In 1968, Montréal beat St. Louis, winning four games in a row. The odds in favour of this occurring were 5 : 4. Determine the probability that Montréal would win any single game in the series. What assumptions did you make?

Lorne "Gump" Worsley played for the Montréal Canadiens, starting in 1963. He won Stanley Cups in 1965, 1966, 1968, and 1969.

19. Gavin is taking two tests. The true-false test has five questions. The multiple-choice test has five questions, each with four choices. Gavin did not study for the tests, so he has to guess each answer. A pass is over 50%. Use a tree diagram to determine the probability that Gavin will pass each test.

Math in Action

Modelling with Probabilities

You can use experimental and theoretical probabilities to solve a mystery.

Without watching, have a classmate put some of the balls (or tiles) in a container. Can you deduce how many of each colour are in the container?

- To arrive at your conjecture, you can run two types of experiments:
  - Draw out one ball, replace the ball, and repeat. Record your results.
  - Draw out two balls, replace the balls, and repeat. Record your results.
- Compare the results of your experiments with theoretical probabilities to make your conjecture.
- Which type of experiment seemed to be the most productive?
- Based on what your group and other groups did, how could you make your conjecture more reliable?
- Is any experiment you could run guaranteed to result in a correct conjecture? Explain.

YOU WILL NEED

- several balls or tiles, in two different colours
- container