

# 6.2

## Characteristics of the Equations of Polynomial Functions

### YOU WILL NEED

- graphing technology

### EXPLORE...

- How can you use the equation of this quadratic function to predict the end behaviour of its graph?

$$f(x) = x^2 - 13x + 40$$

### standard form

The standard form for a linear function is

$$f(x) = ax + b$$

where  $a \neq 0$

The standard form for a quadratic function is

$$f(x) = ax^2 + bx + c$$

where  $a \neq 0$

The standard form

for a cubic function is

$$f(x) = ax^3 + bx^2 + cx + d$$

where  $a \neq 0$

### GOAL

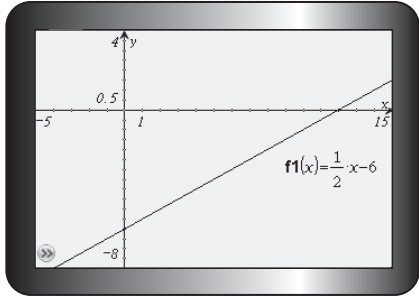
Make connections between the coefficients and constant in the equation of the function and the characteristics of the graph of the function.

### INVESTIGATE the Math

The graph of a linear function can be described from its **standard form** equation, using the slope and the  $y$ -intercept. The graph of a quadratic function can also be described from its standard form equation, using the  $y$ -intercept and the direction of opening.

**?** How can you predict some of the characteristics of the graph of a cubic polynomial function from its standard form equation?

- A.** The equations of nine polynomial functions are given on the next page. Use technology to draw the graph of each function. Use a table like the one below to record the characteristics of each graph.

Standard Form Equation and Graph of Function	
	<p>a) <math>f(x) = \frac{1}{2}x - 6</math></p> 
Degree of Polynomial Function	1
Number of $x$ -Intercepts	1
$y$ -Intercept	-6
End Behaviour	Line extends from quadrant III to quadrant I.
Domain	$\{x \mid x \in \mathbb{R}\}$
Range	$\{y \mid y \in \mathbb{R}\}$
Number of Turning Points	0

- a)  $f(x) = \frac{1}{2}x - 6$   
 b)  $f(x) = -5x - 2$   
 c)  $f(x) = -2x^2 + 2x + 4$   
 d)  $f(x) = x^2 - 6x + 12$   
 e)  $f(x) = -2x^3 + 4x^2 - 3x + 1$   
 f)  $f(x) = 2x^3 + 4x^2 - 3x + 1$   
 g)  $f(x) = x^3 - 2x^2 - 15x + 36$   
 h)  $f(x) = x^3 - 8$   
 i)  $f(x) = -x^3 + 2x^2 + 15x - 10$

- B. How is the **constant term** in a polynomial function related to the  $y$ -intercept of the graph of the function?
- C. How does the sign of the **leading coefficient** affect the end behaviour of the graph of each type of polynomial function?
- D. How can you predict some of the characteristics of the graph of a cubic polynomial function from the standard form equation of the function?

### leading coefficient

The coefficient of the term with the greatest degree in a polynomial function in standard form; for example, the leading coefficient in the function

$$f(x) = 2x^3 + 7x$$

is 2.

## Reflecting

- E. Use a sketch to explain how changing the constant term in a cubic polynomial function can change the number of  $x$ -intercepts on the graph of the function.
- F. Why does the sign of the leading coefficient in a polynomial function affect the end behaviour of the graph?
- G. How does the degree of a polynomial function relate to
- the maximum number of  $x$ -intercepts the graph may have
  - the maximum number of turning points the graph may have

## APPLY the Math

### EXAMPLE 1

Reasoning about the characteristics of the graph of a given polynomial function using its equation

Determine the following characteristics of each function using its equation.

- number of possible  $x$ -intercepts
  - $y$ -intercept
  - end behaviour
  - domain
  - range
  - number of possible turning points
- a)  $f(x) = 3x - 5$   
 b)  $f(x) = -2x^2 - 4x + 8$   
 c)  $f(x) = 2x^3 + 10x^2 - 2x - 10$



## Mary's Solution

a)  $f(x) = 3x - 5$

Degree: 1

Number of  $x$ -intercepts: 1

$y$ -intercept:  $-5$

End behaviour: The graph extends from quadrant III to quadrant I.

Domain:  $\{x \mid x \in \mathbb{R}\}$

Range:  $\{y \mid y \in \mathbb{R}\}$

Number of turning points: 0

-----  
This is a linear polynomial function. Its degree is 1. Polynomial functions of degree 1 have one  $x$ -intercept.

-----  
The constant term of the equation in standard form is  $-5$ , so this is the  $y$ -intercept.

-----  
The function has a positive leading coefficient, so the slope of the line is positive. This means that the line extends from quadrant III to quadrant I.

-----  
The domain and range of a linear function are all real numbers.

-----  
A linear function always increases or always decreases, so it has no turning points.

b)  $f(x) = -2x^2 - 4x + 8$

Degree: 2

Number of  $x$ -intercepts: 2

$y$ -intercept: 8

End behaviour: The graph extends from quadrant III to quadrant IV.

Domain:  $\{x \mid x \in \mathbb{R}\}$

Range:  $\{y \mid y \leq \text{maximum}, y \in \mathbb{R}\}$

-----  
This is a quadratic polynomial function. Its degree is 2.

-----  
Since the function has a negative leading coefficient, I know that the parabola opens down. The constant term is 8, so this is the  $y$ -intercept. Since the  $y$ -intercept is positive and the parabola opens down, I know that the vertex is above the  $x$ -axis. Therefore, the function has two  $x$ -intercepts.

-----  
The parabola opens down, so it extends from quadrant III to quadrant IV.

-----  
The domain is all real numbers. I can't determine the range without knowing the vertex, and I don't know the vertex without graphing the function. From the equation of the function, I know that the parabola opens down. Therefore, the range is restricted to all  $y$ -values less than or equal to the maximum value of the parabola.

c)  $f(x) = 2x^3 + 10x^2 - 2x - 10$

Degree: 3

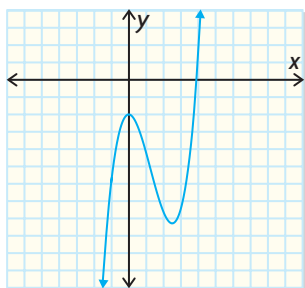
Number of possible  $x$ -intercepts: 1, 2, or 3

-----  
This is a cubic polynomial function. Its degree is 3.

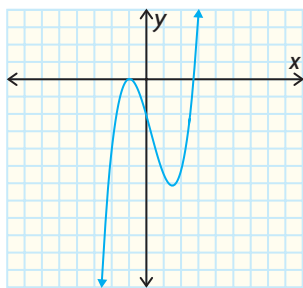
-----  
Depending on the number of turning points and the location of the function on the grid, a polynomial function of degree 3 can have up to three  $x$ -intercepts. I can't determine how many  $x$ -intercepts there are from the equation of the function.



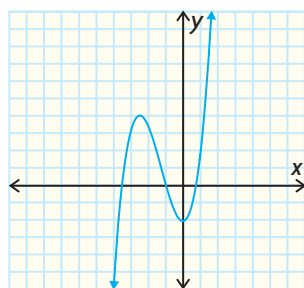
One  $x$ -intercept:



Two  $x$ -intercepts:



Three  $x$ -intercepts:



$y$ -intercept:  $-10$

----- The constant term of the function in standard form is  $-10$ , so this is the  $y$ -intercept.

End behaviour: The graph extends from quadrant III to quadrant I.

----- The function has a positive leading coefficient. The graph of a cubic function extends from quadrant III to quadrant I when the leading coefficient is positive.

Domain:  $\{x \mid x \in \mathbb{R}\}$

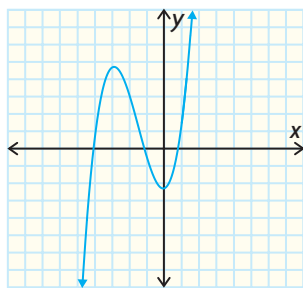
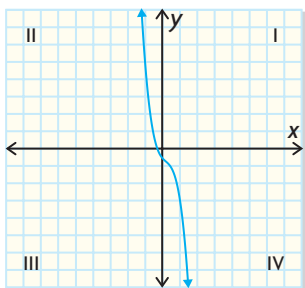
Range:  $\{y \mid y \in \mathbb{R}\}$

----- The domain and range of a cubic function are all real numbers.

Number of possible turning points: 0 or 2

Zero turning points:      Two turning points:

----- A cubic polynomial function can have zero or two turning points. I can't determine how many there are from the equation of the function.



## Your Turn

Change the coefficients of the cubic function in part c) to obtain different end behaviour and a positive  $y$ -intercept. Verify your answer using technology.

**EXAMPLE 2**
**Connecting polynomial functions to their graphs**

Match each graph with the correct polynomial function.

Justify your reasoning.

$$g(x) = -x^3 + 4x^2 - 2x - 2$$

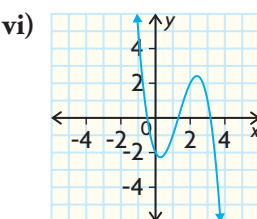
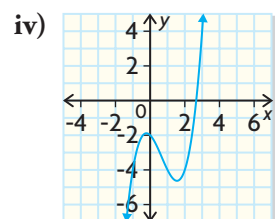
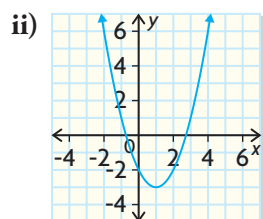
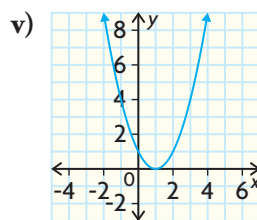
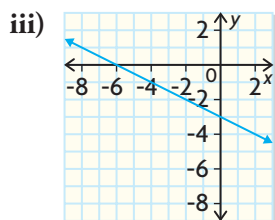
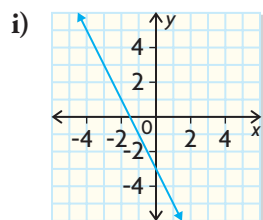
$$j(x) = x^2 - 2x - 2$$

$$p(x) = x^3 - 2x^2 - x - 2$$

$$h(x) = -\frac{1}{2}x - 3$$

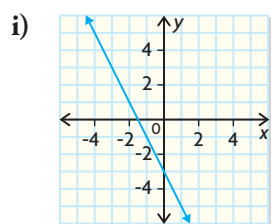
$$k(x) = x^2 - 2x + 1$$

$$q(x) = -2x - 3$$

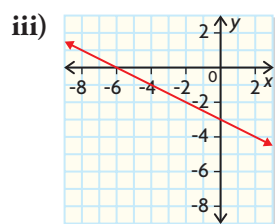

**Jim's Solution**

There are two linear, two quadratic, and two cubic graphs. I grouped together the graphs and equations that represent functions of the same degree.

Linear graphs and equations:



$$q(x) = -2x - 3$$



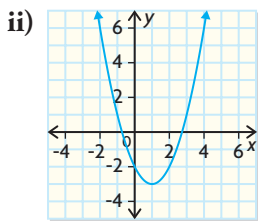
$$h(x) = -\frac{1}{2}x - 3$$

- Both graphs have negative leading coefficients that represent the slope.
- The  $y$ -intercept of both graphs is  $-3$ .
- The difference between the two graphs is the steepness of the lines. Graph i) crosses the  $x$ -axis at  $-1.5$ , and graph iii) crosses the  $x$ -axis at  $-6$ , so the graph for i) is steeper.

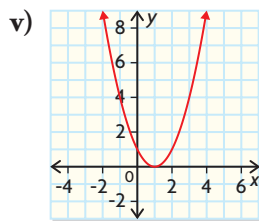
Therefore, graph i) matches  $q(x)$  and graph iii) matches  $h(x)$ .



Parabolic graphs and equations:



$$j(x) = x^2 - 2x - 2$$

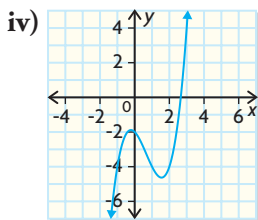


$$k(x) = x^2 - 2x + 1$$

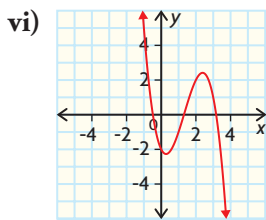
- Both graphs have the same end behaviour, but they have different  $y$ -intercepts.
- Graph ii) has a  $y$ -intercept of  $-2$ , and graph v) has a  $y$ -intercept of  $1$ .  
These  $y$ -intercepts are the constant terms in the equations.

Therefore, graph ii) matches  $j(x)$  and graph v) matches  $k(x)$ .

Cubic graphs and equations:



$$p(x) = x^3 - 2x^2 - x - 2$$



$$g(x) = -x^3 + 4x^2 - 2x - 2$$

- These graphs have the same  $y$ -intercept of  $-2$ , but they have different end behaviour.
- A cubic function with a positive leading coefficient has a graph that extends from quadrant III to quadrant I. Graph iv) extends from quadrant III to quadrant I, so its function has a positive leading coefficient.
- Using similar reasoning, graph vi) extends from quadrant II to quadrant IV, so the function must have a negative leading coefficient.

Therefore, graph iv) matches  $p(x)$  and graph vi) matches  $g(x)$ .

### Your Turn

Describe or draw a sketch to show how each graph in Example 2 would change if the sign of the leading coefficient changed to the opposite sign.

**EXAMPLE 3****Reasoning about the characteristics of the graphs of polynomial functions**

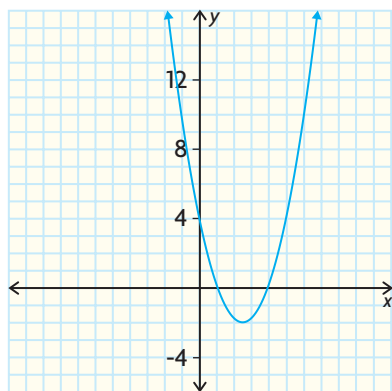
Sketch the graph of a possible polynomial function for each set of characteristics below. What can you conclude about the equation of the function with these characteristics?

- a) Range:  $\{y \mid y \geq -2, y \in \mathbb{R}\}$   
 $y$ -intercept: 4
- b) Range:  $\{y \mid y \in \mathbb{R}\}$   
 Turning points: one in quadrant III and another in quadrant I

**Leanne's Solution**

- a) Range:  $\{y \mid y \geq -2, y \in \mathbb{R}\}$   
 $y$ -intercept: 4  
 Because the range is a subset of the set of real numbers, I know that this function must be a quadratic function. It has a minimum at  $y = -2$ .

----- Since the quadratic function has a minimum, the parabola opens upward.



----- My sketch satisfies the given characteristics, but I know that there are many other possibilities. For example, the vertex could have been in quadrant III.

Because of the end behaviour in my sketch, I know that the equation of the function has a positive leading coefficient.

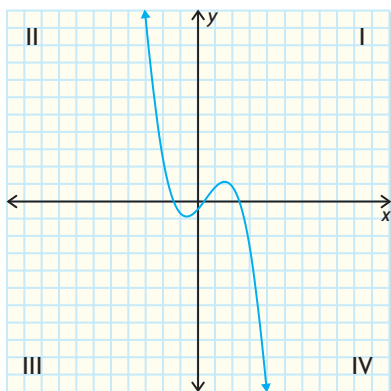
I also know that the constant term is 4, because this is the  $y$ -intercept of the graph.



b) Range:  $\{y \mid y \in \mathbb{R}\}$

Turning points: one in quadrant III and another in quadrant I

The range is all real numbers, so I know that this is a linear or cubic function. There are two turning points, so I know that it is a cubic polynomial function.



I tried sketching different cubic functions with turning points in quadrants III and I, but I noticed that they all had the same shape. I was only able to sketch a cubic function extending from quadrant II to quadrant IV with the given characteristics.

Since the graph of the function extends from quadrant II to IV, the equation must have a negative leading coefficient.

I was not able to determine the  $y$ -intercept using the characteristics given, so I do not know what the constant term is. The curve could be on the origin, or it could pass above or below the origin.

### Your Turn

Write a possible equation for each function above. Use technology to check and adjust your equation so that it satisfies the given characteristics.



## In Summary

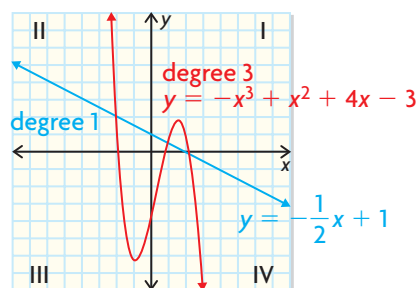
### Key Ideas

- When a polynomial function is in standard form:
  - The maximum number of  $x$ -intercepts the graph may have is equal to the degree of the function.
  - The maximum number of turning points a graph may have is equal to one less than the degree of the function.
  - The degree and leading coefficient of the equation of a polynomial function indicate the end behaviour of the graph of the function.
  - The constant term in the equation of a polynomial function is the  $y$ -intercept of its graph.

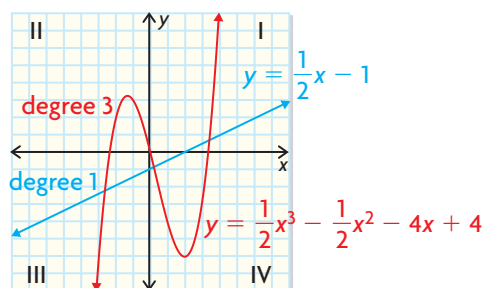
### Need to Know

- Linear and cubic polynomial functions with positive leading coefficients have similar end behaviour. Linear and cubic polynomial functions with negative leading coefficients also have similar end behaviour.

If the leading coefficient is negative, then the graph of the function extends from quadrant II to quadrant IV.

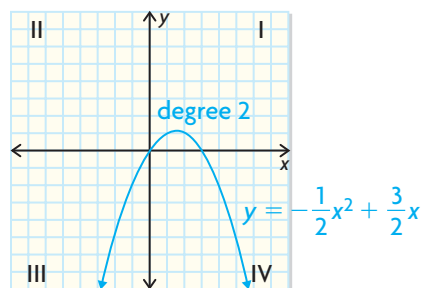


If the leading coefficient is positive, then the graph of the function extends from quadrant III to quadrant I.

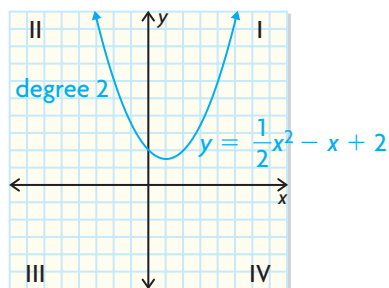


- Quadratic polynomial functions have unique end behaviour.

If the leading coefficient is negative, then the graph of the function extends from quadrant III to quadrant IV.

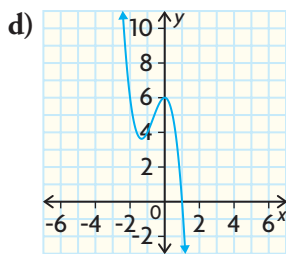
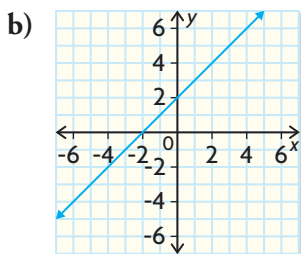
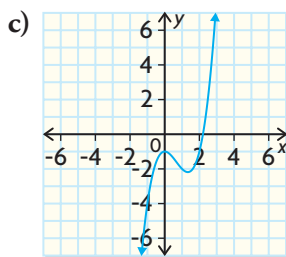
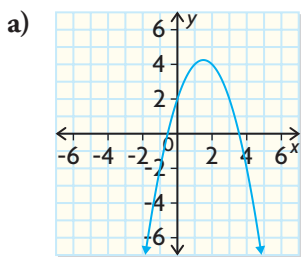


If the leading coefficient is positive, then the graph of the function extends from quadrant II to quadrant I.



## CHECK Your Understanding

- Determine the degree, the leading coefficient, and the constant term for each polynomial function.
  - $f(x) = 6x^2 - 3x - 2$
  - $g(x) = -\frac{2}{3}x + 10$
  - $h(x) = -x^3 + 10x + 6$
  - $j(x) = 4x^3 - 2x^2 - 3x - 10$
- For each function in question 1:
  - Determine the minimum and maximum number of  $x$ -intercepts of the graph.
  - Determine the end behaviour, domain, and range of the graph.
  - Determine the minimum and maximum number of turning points on the graph.
- Determine the degree, the sign of the leading coefficient, and the constant term for the polynomial function represented by each graph below.



## PRACTISING

- For each type of polynomial function below, write an equation that has a  $y$ -intercept of 5.
  - constant
  - linear
  - quadratic
  - cubic

5. Describe the end behaviour of each polynomial function.

a)  $f(x) = 3x^3 - 2x^2 + x - 1$

b)  $g(x) = -x^2 + x + 5$

c)  $h(x) = x(x + 2)$

d)  $p(x) = 5x + 6 - x^3$

e)  $q(x) = -x - 1$

f)  $r(x) = \frac{3}{2}x^3 - 2x^2$

6. Match each graph with the correct polynomial function. Justify your reasoning.

i)  $y = -x^3 + x + 4$

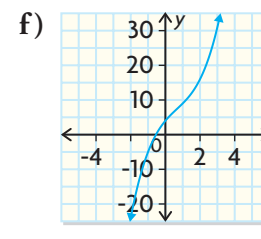
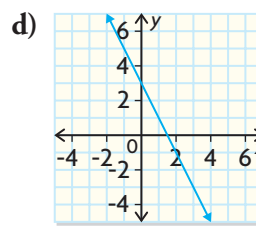
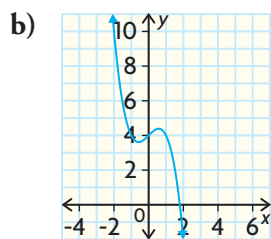
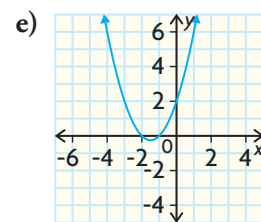
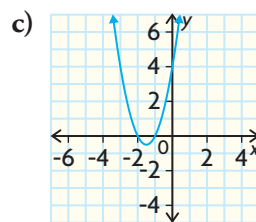
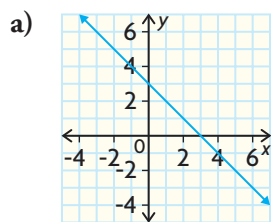
iv)  $y = x^3 - 2x^2 + 6x + 4$

ii)  $y = 2x^2 + 6x + 4$

v)  $y = 3 - x$

iii)  $y = (x + 1)(x + 2)$

vi)  $y = -2x + 3$



7. Determine the following characteristics of each function.

- number of possible  $x$ -intercepts
- $y$ -intercept
- end behaviour
- domain
- range
- number of possible turning points

a)  $f(x) = -2x + 5$

b)  $v(x) = x^2 + 2x - 6$

c)  $u(x) = x^3 - x^2 + 5x - 1$

d)  $w(x) = -2x^3 + 4x$

8. Write an equation for a polynomial function that satisfies each set of characteristics.

- a) extending from quadrant III to quadrant IV, one turning point,  $y$ -intercept of 2
- b) extending from quadrant III to quadrant I, three  $x$ -intercepts
- c) degree 1, increasing function,  $y$ -intercept of  $-3$
- d) two turning points,  $y$ -intercept of 5
- e) range of  $y \leq 2$ ,  $y$ -intercept of 2

9. Lukas described the characteristics of the graph of a polynomial function, but he made an error. Explain and correct Lukas's error.

**Lukas's Solution**

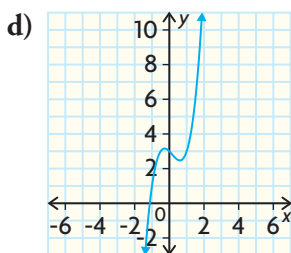
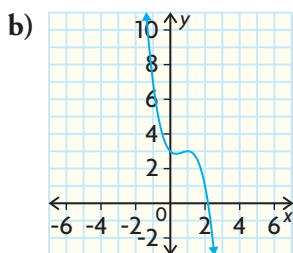
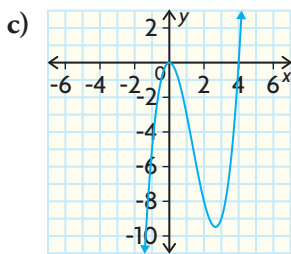
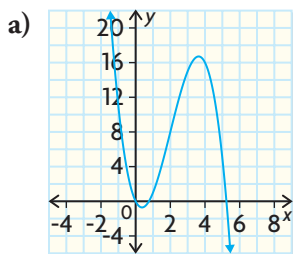
$$f(x) = -3x^3 + 5x^2 + 11x$$

This is a polynomial function of degree 3, so I know that the graph will have zero or two turning points.

The leading coefficient of this cubic function is negative, so I know that the graph extends from quadrant II to quadrant IV.

There is no constant term, so there is no  $y$ -intercept. Because this function is cubic, I know that there can be 1, 2, or 3  $x$ -intercepts.

10. Sketch two possible graphs of polynomial functions that satisfy each set of characteristics.
- degree 2, one turning point which is a maximum, constant term of  $-6$
  - two turning points, positive leading coefficient, one  $x$ -intercept
  - degree 1, negative leading coefficient, constant term of 10
  - cubic, three  $x$ -intercepts, negative leading coefficient
11. Explain why cubic functions may have one, two, or three  $x$ -intercepts. Use sketches to support your explanation.
12. Explain why quadratic polynomial functions have maximum or minimum values, but cubic polynomial functions have only turning points.
13. Match each graph with the correct polynomial function. Justify your reasoning.
- $f(x) = -x^3 + 6x^2 - 4x$
  - $g(x) = 2x^3 - x^2 - x + 3$
  - $h(x) = -x^3 + 2x^2 - x + 3$
  - $j(x) = x^3 - 4x^2$



## Math in Action

### Motion Due to Gravity

For any object that free falls, the graph of height over time is a parabola. The height of an object,  $h$ , in metres, after  $t$  seconds can be expressed as  $h(t) = -4.9t^2 + h_0$  where the constant term,  $h_0$ , represents the initial height of the object.

- In a small group, devise an experiment to determine the height of a window from the ground outside using drops of water.
- Use the leading coefficient, the degree, and the constant term to describe the characteristics of a polynomial function that can model a falling drop of water.
- Find a window that is high above the ground, and carry out your experiment.

14. The average retail price of gas in Canada, from 1979 to 2008, can be modelled by the polynomial function

$$P(y) = 0.008y^3 - 0.307y^2 + 4.830y + 25.720$$

where  $P$  is the price of gas in cents per litre and  $y$  is the number of years after 1979.

- Describe the characteristics of the graph of the polynomial function.
- Explain what the constant term means in the context of this problem.

15. The tide depth in Deep Cove, British Columbia, from 04:00 to 15:00 on January 6, 2011, can be modelled by the polynomial function

$$f(t) = 0.001t^3 - 0.055t^2 + 0.845t + 0.293$$

where  $f(t)$  is the tide depth in metres and  $t$  is the number of hours after midnight.

- Use the polynomial function to determine the tide depth at 10:00 on January 6. Compare your result with the actual tide depth in the table to the left.
- Describe the end behaviour of this function.
- Does this function provide accurate tide depths outside the given time frame? Check using the data provided for Jan. 10, 2011.

Time of Day	Tide Depth (m)	
	Jan. 6, 2011	Jan. 10, 2011
00:00	0.8	2.6
01:00	0.8	2.3
02:00	1.2	2.0
03:00	1.8	2.0
04:00	2.6	2.3
05:00	3.5	2.7
06:00	4.2	3.3
07:00	4.7	3.8
08:00	4.8	4.3
09:00	4.6	4.6
10:00	4.2	4.6
11:00	3.8	4.3
12:00	3.5	3.9
13:00	3.2	3.4
14:00	3.2	3.0
15:00	3.4	2.6
16:00	3.6	2.3
17:00	3.8	2.3
18:00	3.9	2.4
19:00	3.8	2.7
20:00	3.4	3.0
21:00	2.9	3.2
22:00	2.2	3.2
23:00	1.6	3.2

Statistics Canada



## Closing

16. Suppose that you wanted to describe the graph or equation of a polynomial function and were allowed to ask only three questions about the function. What questions would you ask? Why?

## Extending

17. a) Use technology to determine the locations of the turning points of each polynomial function.
- i)  $f(x) = x(x - 2)(x - 4)$
  - ii)  $g(x) = (x - 6)(x - 2)(x + 1)$
  - iii)  $h(x) = -x(x + 3)(x + 6)$
- b) Sketch the graph of each function.
- c) Explain how you could approximate the locations of the turning points without technology.
18. The probability of sinking exactly two out of three free throws with a basketball is given by the function

$$P(x) = 3x^2 - 3x^3$$

where  $x$  is the probability of sinking one throw.

- a) Check the reasonableness of the function by evaluating it for different probabilities.
- b) Determine the domain and range of the function.
- c) Determine the coordinates of the turning point of the function within the domain. Explain the meaning of the turning point in the context of this problem.
- d) Determine the  $x$ -intercepts, and explain their meaning in the context of this problem.

