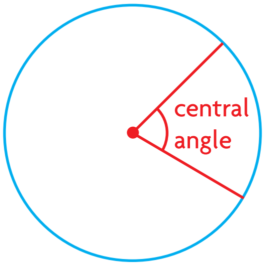
8.1 Understanding Angles

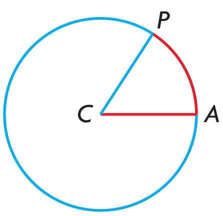
YOU WILL NEED

- compass
- pipe cleaners
- protractor

EXPLORE...

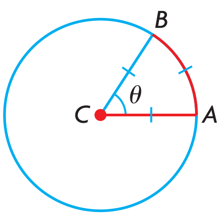
- Devise a method to estimate the measure of any central angle in a circle without using a protractor. Explain how your method works.





radian

The measure of the central angle of a circle subtended by an arc that is the same length as the radius of the circle.



$\angle\theta = 1$

GOAL

Estimate and determine benchmarks for angle measure.

INVESTIGATE the Math

Abner is a carpenter. When he needs to cut a hole in wood, he attaches a hole saw to a drill. For one project, he used hole saws with diameters of 10 cm, 14 cm, and 20 cm. Abner knows that when the diameter of the saw increases, the circumference of the hole also increases.

? How many times will the radius of each hole saw fit around the circumference of the circle it cuts?

A. Work in groups of three, with each member drawing a circle to represent one of the three different hole saws. Label the radius of your circle CA.

B. Cut a pipe cleaner to a length equal to radius CA.

C. Place and bend the pipe cleaner around the circumference of your circle, beginning at point A and ending at point P, as shown by the red arc.

D. Draw radius CP.

E. Measure $\angle PCA$ with a protractor. What is the measure of $\angle PCA$ in degrees? Is the measure of $\angle PCA$ the same for all three circles?

F. Use the pipe cleaner to continue marking radius lengths all the way around your circle. About how many radius lengths are there in one complete circumference of your hole saw? Compare your results with those of your group members.

Reflecting

G. The circumference, C, of a circle is given by the formula

$$C = 2\pi r$$

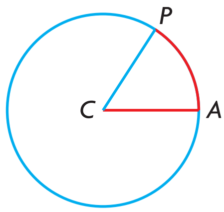
where r represents the radius. Explain how this formula relates to your answer for part F.

H. What degree measure is equal to 2π **radians**?

APPLY the Math

EXAMPLE 1 Expressing 1 radian in degrees

The measure of $\angle PCA$ is 1 radian. Calculate the measure of $\angle PCA$ in degrees.



> **Communication *Tip***
>
> Angles can be measured using different units. These include degrees, radians, gradients, and minutes and seconds.

Thao's Solution

One complete revolution of the circle measures 360°.
One complete revolution of the circle also measures 2π in radians.

$$2\pi \text{ radians} = 360°$$

> I knew the total number of degrees in a circle and the total number of radians in a circle.
>
> I determined the relationship between degrees and radians using the transitive property.

So π radians must be equal to $\frac{360°}{2}$.

$$\pi \text{ radians} = 180°$$

To determine the measure of 1 radian,
I can divide both sides of the equation by π:

$$1 \text{ radian} = \frac{180°}{\pi}$$

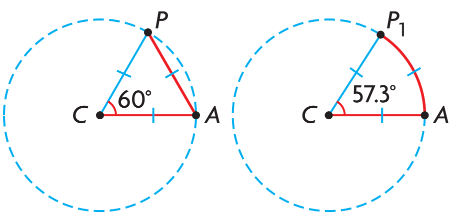
$$1 \text{ radian} = 57.295\ldots°$$

$\angle PCA$ measures about 57.3°.

> The value of π is 3.141…, and 180° is equivalent to π, so I know that 180° is equivalent to just over 3 radians. So it makes sense that 1 radian is just under 60°.

To visualize 1 radian, use an equilateral triangle with a flexible side, *AP*, drawn inside a circle.

> I imagined moving the point *P* along the circumference of the circle, forcing the chord *AP* to bend, until it fit along the circumference, from point *A* to point P_1.



> Since side *PC* is moved toward side *AC*, the angle in the second diagram must measure a bit less than 60°.

Therefore, 60° can be used as a benchmark for 1 radian.

Your Turn

Suggest two more benchmarks you could use to estimate the size of an angle given in radian measure.

EXAMPLE 2 Estimating values of angles in radian measure

Estimate the value of each angle in radian measure.

a) 90° **b)** 45° **c)** 150°

Natalia's Solution

a) 90° is half of 180°.
Since 180° is equivalent to π in radian measure, I can use this relationship to estimate 90° in radians. I can round π to 3.2 so I can do mental calculations more easily.

I decided to relate 90° to the benchmark angle of 180°.

I can estimate 90° as $\frac{3.2}{2}$, or 1.6.

Communication Tip
Any angle measures presented as real numbers without units are considered to be in radians.

b) 45° is half of 90°, so 45° must be about half of 1.6.
I can estimate 45° as $\frac{1.6}{2}$, or 0.8.

I divided by 2 again to determine my estimate.

I knew that my estimate was high, because I rounded π up to 3.2.

c) I can use the benchmark that 1 radian is slightly less than 60° to estimate the measure of 30° in radians.

150° is 30° less than 180°.
I can estimate 30° as about 0.5 radians.

I can round π to 3.2 so I can easily do mental calculations.

180° is equivalent to π in radian measure.
I can estimate 150° as $3.2 - 0.5$, or 2.7.

I knew that my estimate was a bit high, because I rounded π up to 3.2.

Your Turn

Estimate the value of each angle in radian measure.

a) 120° **b)** 135°

Communication Tip
It is possible to get different estimates of angle measures. Your estimates may vary depending on which benchmarks you use.

EXAMPLE 3 Estimating angles greater than 180° in radian measure

Estimate the value of each angle in radian measure.

a) 240° **b)** 450° **c)** 690°

Sheila's Solution

a) I thought of 240° as the sum of 180° and 60°.
180° is slightly less than 3.2 radians.
60° is slightly more than 1 radian.
I estimate 240° is equivalent to about 4.2 radians.

> I knew that 1 radian is slightly less than 60°.

b) I knew that 450° is 90° more than 360°.
360° is about 6.3 radians.
90° is about 1.6 radians.
I estimate 450° is slightly less than 7.9 radians.

> I knew that my estimate was slightly high, because I rounded up the radian equivalents for both 360° and 90°.

c) Two complete revolutions of the circle measure 720°.
690° is 30° less than 720°.
30° is half of 60°, so it is about 0.5 in radian measure.
Two complete revolutions are about $2 \cdot 6.3$, or 12.6 radians.
690° is about $12.6 - 0.5$, or 12.1 radians.

> I knew that 60° is about 1 in radian measure.

> I knew that 1 complete revolution measures 2π radians, which is about 6.3.

> I estimated 30° in radian measure. Then I subtracted this value from two revolutions.

Your Turn

Estimate the value of each angle in radian measure.

a) 420° **b)** 495° **c)** 660°

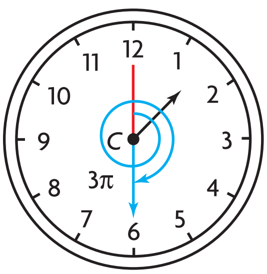
EXAMPLE 4 Comparing angles in radian measure

Determine which angle is larger: 3π or 8.

Xavier's Solution: Using benchmarks and visualization

3π is equivalent to $1\frac{1}{2}$ revolutions.

> I knew that π represents $\frac{1}{2}$ revolution and 2π represents 1 revolution.



> I visualized $1\frac{1}{2}$ revolutions using the minute hand of a clock. Starting at noon, $1\frac{1}{2}$ revolutions is equivalent to 1:30.

1 revolution is about 6.3 in radian measure.

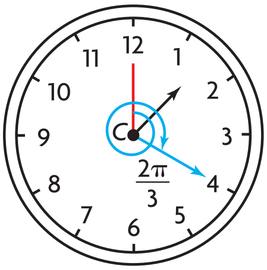
$\frac{1}{4}$ revolution is about 1.6 radians.

Therefore, $1\frac{1}{4}$ revolutions is about 7.9 radians.

I used benchmarks to estimate the position of the second angle, 8. I knew that $\frac{1}{2}$ revolution, 180°, is π in radian measure, or about 3.2. I divided by 2 to get an estimate for $\frac{1}{4}$ revolution, 90°.
I knew that my estimate for $1\frac{1}{4}$ revolutions was slightly high, because I rounded up my radian estimates for both 1 revolution and $\frac{1}{4}$ revolution.

8 is a little greater than 7.9.

So, 8 is a little greater than $1\frac{1}{4}$ revolutions.



I visualized an angle that is a little greater than $1\frac{1}{4}$ revolutions on a clock, which is equivalent to 1:15 if I started at noon. The minute hand should be close to about 4.

$1\frac{1}{2}$ revolutions is greater.

Therefore, 3π is larger than 8.

My answer makes sense since 3π is about 3(3.14), or 9.42, in radian measure.

Zachary's Solution: Expressing the angles in degrees

$\pi = 180°$
$2\pi = 360°$

I knew the degree equivalents for π and 2π radians.

$3\pi = 180° + 360°$
$3\pi = 540°$

8 is equivalent to about $8 \cdot 60°$, or 480°.

I knew that 1 in radian measure is about 60°, so 8 in radian measure is about 8 times 60°.

I knew that my estimate was high, because 1 is slightly less than 60°.

$540° > 480°$
3π is larger than 8.

Your Turn

Which solution do you prefer: Xavier's or Zachary's? Explain.

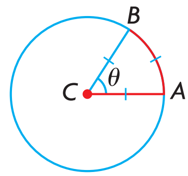
In Summary

Key Ideas

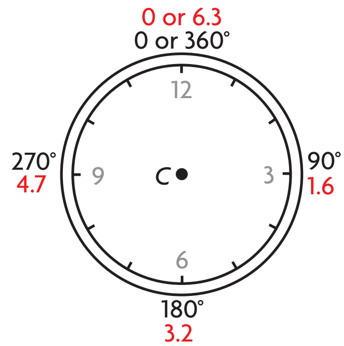
- Radian measure is an alternative way to express the size of an angle.
- Using radians allows you to express the measure of an angle as a real number without units.
- The central angle formed by one complete revolution in a circle is 360°, or 2π in radian measure.

Need to Know

- Use benchmarks to estimate the degree measure of an angle given in radians.
- In radian measure,
  - 1 is equivalent to about 60°;
  - π is equivalent to 180°;
  - 2π is equivalent to 360°.
- Decimal approximations can be used for benchmarks to visualize the approximate size of an angle measured in radians.



$\angle\theta = 57.3°$ or 1 in radian measure



Some estimation benchmarks

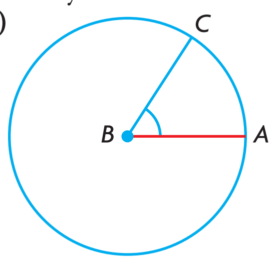
CHECK Your Understanding

1. Sketch an angle with each degree measure, and then estimate the measure in radians. Estimate to the nearest tenth.
   a) 45° **b)** 150° **c)** 180°
2. Estimate the value of each radian measure in degrees. Estimate to the nearest degree.
   a) 1.6 **b)** 0.5 **c)** 2.4 **d)** 4.7

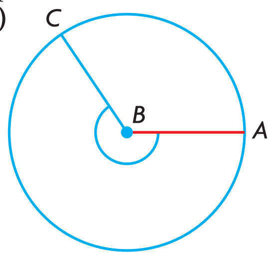
PRACTISING

3. Estimate, to the nearest degree, the measure of each central angle. Check your estimate with a protractor.

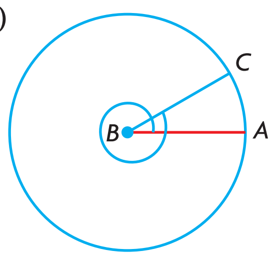
a)



b)



c)



4. Sketch an angle with each given measure, and then estimate, to the nearest tenth, the equivalent measure in radians.

a) 400° **b)** 640°

5. Estimate, to the nearest degree, the equivalent measure in degrees.

a) 8.1 **b)** 10.5

6. Imagine that it is now 9 a.m.

a) What time will it be when the minute hand has rotated through each of the following angles?

i) 120° **ii)** 330° **iii)** 690°

b) Estimate, to the nearest tenth, each angle measure in radian measure.

7. Imagine that you are standing on the circumference of a circle that has a radius of 3 m. You move a third of the way around the circle.

a) How far did you travel?

b) What central angle was created? Express the measure in degrees and radians.

8. For each pair of angle measures, determine which measure is greater.

a) 100°, 2 **b)** 45°, 0.5 **c)** 280°, 5 **d)** 400°, 6.5

9. Nim claims that any central angle, measured in radians, in a circle with a radius of 5 m will be half the measure of an equivalent angle in a circle with a radius of 10 m. Do you agree or disagree? Explain.

Closing

10. Explain how you could estimate the radian measure equivalent of an angle greater than 360°. Give three examples. Provide two different estimation strategies for each example.

Extending

11. Sketch an angle with each measure.

a) 1200° **b)** 29.3

12. Estimate, to the nearest tenth, the measure of each central angle in radians. Check your estimate by measuring the angle with a protractor.