

6.3

Graphing to Solve Systems of Linear Inequalities

YOU WILL NEED

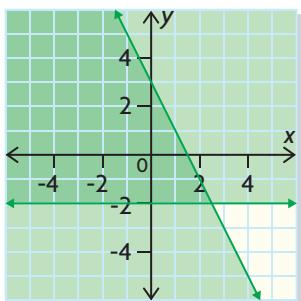
- graphing technology OR graph paper, ruler, and coloured pencils

GOAL

Solve problems by modelling systems of linear inequalities.

EXPLORE...

- What conclusions can you make about the system of linear inequalities graphed below?



LEARN ABOUT the Math

A company makes two types of boats on different assembly lines: aluminum fishing boats and fibreglass bow riders.

- When both assembly lines are running at full capacity, a maximum of 20 boats can be made in a day.
- The demand for fibreglass boats is greater than the demand for aluminum boats, so the company makes at least 5 more fibreglass boats than aluminum boats each day.



- ② What combinations of boats should the company make each day?

EXAMPLE 1

Solving a problem with discrete whole-number variables using a system of inequalities

Mary's Solution: Using graph paper

Let a represent the number of aluminum fishing boats.
Let f represent the number of fibreglass bow riders.

$$a \in \mathbb{W} \text{ and } f \in \mathbb{W}$$

The relationship between the two types of boats can be represented by this system of inequalities:

$$\begin{aligned}a + f &\leq 20 \\a + 5 &\leq f\end{aligned}$$

I knew I could solve this problem by representing the situation algebraically with a system of two linear inequalities and graphing the system.

Since only complete boats are sold, I knew that a and f are whole numbers and the graph would consist of discrete points in the first quadrant.

The two inequalities describe

- a combination of boats to a maximum of 20.
- at least 5 more fibreglass boats than aluminum boats.



$$a + f = 20$$

f -intercept:

$$0 + f = 20$$

$$f = 20$$

$$(0, 20)$$

a -intercept:

$$a + 0 = 20$$

$$a = 20$$

$$(20, 0)$$

$$a + 5 = f$$

f -intercept:

$$0 + 5 = f$$

$$f = 5$$

$$(0, 5)$$

a -intercept:

$$a + 5 = 0$$

$$a = -5$$

$$(-5, 0)$$

$$a + 5 = f$$

$$(5) + 5 = f$$

$$10 = f$$

(5, 10) is a point on this boundary.

Test (0, 0) in $a + f \leq 20$.

LS	RS
$a + f$	20
$0 + 0$	
0	

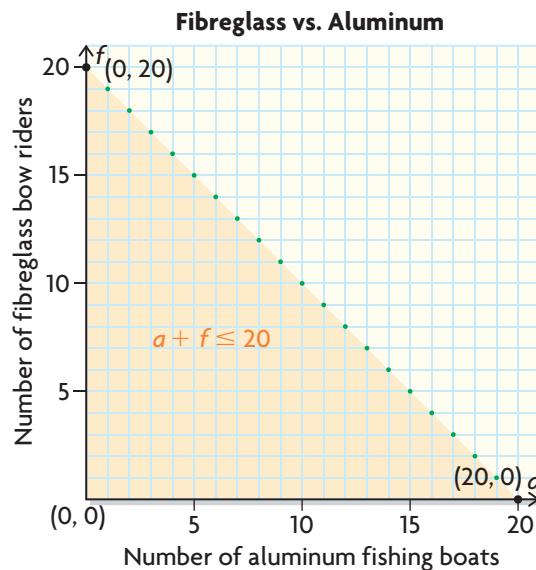
Since $0 \leq 20$, (0, 0) is in the solution region.

To graph each linear inequality, I knew I had to graph its boundary as a stippled line, and then shade and stipple the correct half plane.

To graph each boundary, I wrote each linear equation and then determined the a - and f -intercepts so I could plot and join them.

For $a + 5 = f$, I knew $(-5, 0)$ wasn't going to be a point on the boundary, because it's not in the first quadrant, so I chose another point by solving the equation for $a = 5$.

I tested point (0, 0) to determine which half plane to shade for $a + f \leq 20$.



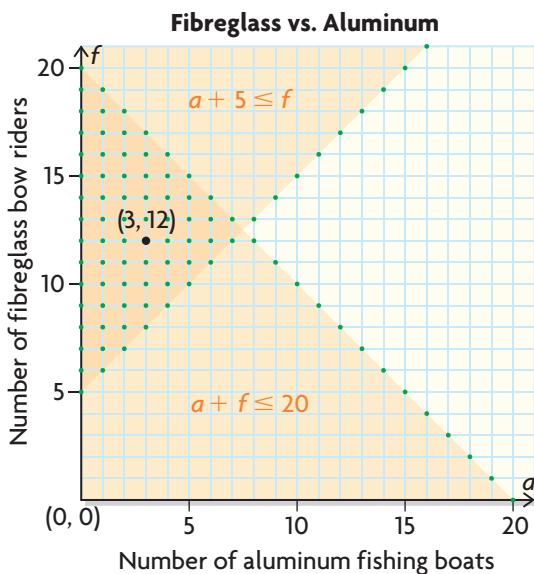
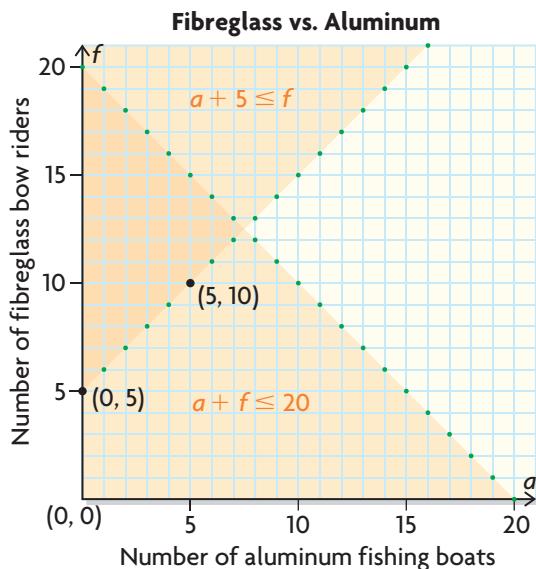
I drew a green stippled boundary connecting (0, 20) and (20, 0) and shaded the half plane below it orange, because the solution region is discrete.

Test (0, 0) in $a + 5 \leq f$.

LS	RS
$a + 5$	f
$0 + 5$	0
5	

Since 5 is not less than or equal to 0, (0, 0) is not in the solution region.

I tested (0, 0) to determine which half plane to shade for $a + 5 \leq f$.



$$\{(a, f) \mid a + f \leq 20, a \in \mathbb{W}, f \in \mathbb{W}\}$$

$$\{(a, f) \mid a + 5 \leq f, a \in \mathbb{W}, f \in \mathbb{W}\}$$

Any point with whole-number coordinates in the intersecting or overlapping region is an acceptable combination. For example, 3 aluminum boats and 12 fibreglass boats is an acceptable combination.

I plotted the points $(0, 5)$ and $(5, 10)$ on the same coordinate plane. I used these points to draw a green stippled boundary for $a + 5 \leq f$.

I shaded the half plane above the boundary orange, since the test point $(0, 0)$ is not a solution to the linear inequality and the solution region is discrete.

I knew that the solution set for the system of linear inequalities is represented by the intersection or overlap of the solution regions of the two inequalities. This made sense since points in this region satisfy both inequalities.

I knew that the triangular solution region included discrete points along its three boundaries, including the y -axis from $y = 5$ to $y = 20$.

Since the solution set for the system contains only discrete points with whole-number coordinates, I stippled its solution region.

I knew that any whole-number point in the triangular solution region is a possible solution. For example, $(3, 12)$ is a possible solution.

I knew that $(3, 12)$ worked because this gives a total of 15 boats with 9 more fibreglass boats than aluminum boats.

Liv's Solution: Using graphing technology

Let x represent the number of aluminum fishing boats.

Let y represent the number of fibreglass bow riders.

$$x \in \mathbb{W} \text{ and } y \in \mathbb{W}$$

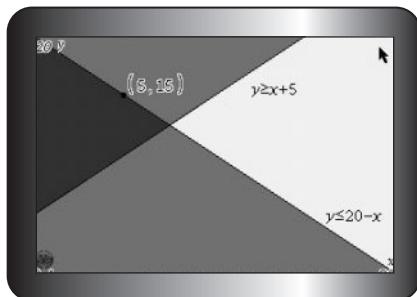
The relationship between the two types of boats can be represented by this system of inequalities:

$$x + y \leq 20$$

$$x + 5 \leq y$$

$$\begin{aligned}x + y &\leq 20 \\y &\leq 20 - x \\y &\leq -x + 20\end{aligned}$$

$$\begin{aligned}x + 5 &\leq y \\y &\geq x + 5\end{aligned}$$



Any point with whole-number coordinates in the intersecting or overlapping solution region is an acceptable combination. For example, 5 aluminum boats and 15 fibreglass boats is an acceptable combination:

$$x + y \leq 20$$

$$(5) + (15) \leq 20$$

$$20 \leq 20$$

This is valid.

$$x + 5 \leq y$$

$$(5) + (5) \leq 15$$

$$10 \leq 15$$

This is valid.

I knew that I could solve this problem by representing it algebraically with a system of two linear inequalities and graphing the system.

Since boats are made in units, I knew that x and y are whole numbers and the graph would be in the first quadrant.

The two inequalities describe

- a combination of boats to a maximum of 20.
- at least 5 more fibreglass boats than aluminum boats.

I rearranged each linear inequality to isolate y , so I could enter the system into the graphing calculator.

Before graphing, I set my calculator window to show only the first quadrant, with maximum x - and y -values of 20 because only whole numbers of boats can be made and 20 is the maximum.

When interpreting the graph, I was careful to think about the problem.

- I knew only points with whole-number coordinates in the intersecting or overlapping solution region are valid.
- I knew that the triangular solution region includes points with whole-number coordinates along its three boundaries: the two solid line boundaries and the y -axis from $y = 5$ to $y = 20$.

I chose point $(5, 15)$ on the boundary with whole-number coordinates as a possible solution, and I checked to see if it satisfied both inequalities.

Reflecting

- Is every point on the boundaries of the solution region a possible solution? Explain.
- Are the three points where the boundaries intersect part of the solution region? Explain.
- How would the graph change if fewer than 25 boats were made each day?
- All points with whole-number coordinates in the solution region are valid, but are they all reasonable? Explain.

APPLY the Math

EXAMPLE 2

Solving graphically a system of two linear inequalities with continuous variables

Graph the solution set for the following system of inequalities. Choose two possible solutions from the set.

$$3x + 2y > -6$$

$$y \leq 3$$

Peter's Solution: Using graph paper

$$x \in \mathbb{R}, y \in \mathbb{R}$$

$$3x + 2y > -6$$

$$x\text{-intercept:}$$

$$3x + 2(0) = -6$$

$$\frac{3x}{3} = -\frac{6}{3}$$

$$x = -2$$

$$(-2, 0)$$

$$y\text{-intercept:}$$

$$3(0) + 2y = -6$$

$$\frac{2y}{2} = -\frac{6}{2}$$

$$y = -3$$

$$(0, -3)$$

I assumed both x and y are in the set of real numbers because restrictions on the domain and range were not stated. I knew the graph would have a continuous solution region and could be in all four quadrants.

To graph $3x + 2y > -6$, I identified the x - and y -intercepts of the linear equation of the boundary $3x + 2y = -6$.

Test $(0, 0)$ in $3x + 2y > -6$.

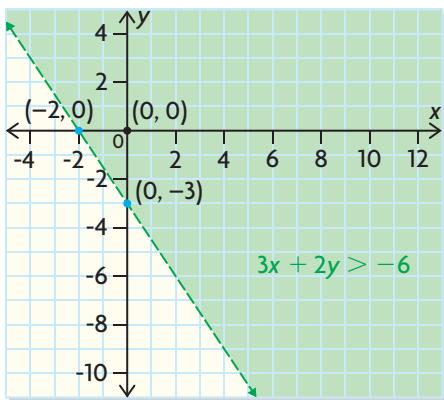
LS	RS
$3x + 2y$	
$3(0) + 2(0)$	-6
0	

I used the test point $(0, 0)$ to determine which region to shade.

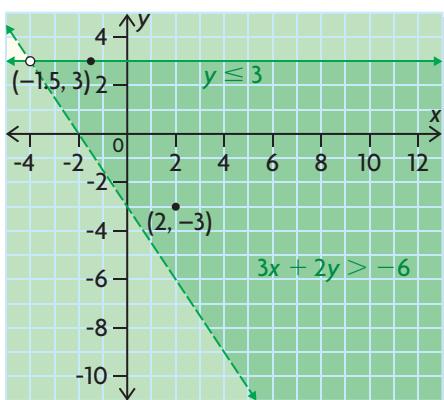
Since $0 > -6$,

$(0, 0)$ is in the solution region.





$$y \leq 3$$



The overlapping solution region represents the solution set of the system of linear inequalities. Therefore, $(2, -3)$ and $(-1.5, 3)$ are two possible solutions.

I drew a dashed green line for the boundary since the $>$ sign does not include the possibility of equality and the solution set is continuous.

I shaded the half plane that included $(0, 0)$, since $(0, 0)$ is a solution to the linear inequality. I used green shading to show a continuous solution region.

I knew that I should draw a solid horizontal green boundary because the inequality has one variable, y , the sign is \leq and the solution set is continuous.

I shaded the half plane below the boundary, since all the points in this region have y -coordinates that are less than 3.

Where the solid and dashed boundaries intersect, I drew an open dot to show that this point is not part of the solution region. It made sense that the intersection point is not included because none of the points on the boundary of $3x + 2y > -6$ are included in its solution region.

I knew that all the points in the overlapping solution region, which included points along its solid boundary, represented the solution set, because x and y are in the set of real numbers.

Any point in the solution region is a possible solution.

Rita's Solution: Using graphing technology

$$x \in \mathbb{R}, y \in \mathbb{R}$$

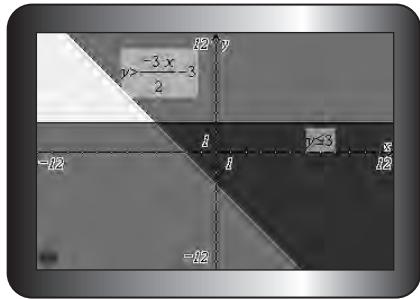
$$y \leq 3$$

$$3x + 2y > -6$$

$$\begin{aligned} \frac{2y}{2} &> -\frac{3x}{2} - \frac{6}{2} \\ y &> -\frac{3x}{2} - 3 \end{aligned}$$

I assumed both x and y are in the set of real numbers because no restrictions were stated. I knew the graph would have a continuous solution region and could be in all four quadrants.

To enter the inequalities into my graphing calculator, I isolated y .



The overlapping solution region represents the solution set of the system of inequalities.

I set my calculator window to show all four quadrants, since $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

I entered $y > -\frac{3x}{2} - 3$ and $y \leq 3$, and then graphed the system.

I interpreted the overlapping region as follows:

- The solution region is where the solution regions of the two inequalities intersect.
- Only solid boundaries are included in the solution region.
- The point where the dashed line intersects the solid line is not part of the solution.
- All the points in the solution region represent the solution set for the system because x and y are real numbers.

Any point in the solution region is a possible solution.

Therefore, $(2, 2)$ and $(-0.5, -1.7)$ are two possible solutions.

Your Turn

How would the solution region change if $x \in \mathbb{I}$ and $y \in \mathbb{I}$?

How would it stay the same?

EXAMPLE 3**Solving graphically a problem with continuous positive variables**

A sloop is a sailboat with two sails: a mainsail and a jib. When a sail is fully out or up, it is said to be “out 100%.” When the winds are high, sailors often reef, or pull in, the sails to be less than their full capability.

- Jim is sailing in winds of 22 knots, so he wants no more than 80% of the mainsail out.
- Jim also wants more mainsail out than jib.

What possible combinations of mainsail and jib can Jim have out?

**Louise's Solution: Using graph paper**

Let m represent the percent of mainsail out.

Let j represent the percent of jib out.

$$m \geq 0 \text{ and } j \geq 0, \text{ where } m \in \mathbb{R}, j \in \mathbb{R}$$

I knew that I could solve the problem by representing it algebraically with a system of two linear inequalities and then graphing it.

I knew that the graph would be in the first quadrant since there can't be negative percents of sails out.

I also knew that the solution region would be continuous since decimal percents are possible.

The relationship between the two types of sails can be represented by the following system of two linear inequalities:

$$m \leq 80$$

$$j < m$$

$$m \leq 80$$

Boundary: $m = 80$

Boundary is a vertical line with an m -intercept of 80.

$$j < m$$

Boundary: $j = m$

Boundary line has a slope of 1 and a j -intercept of 0.

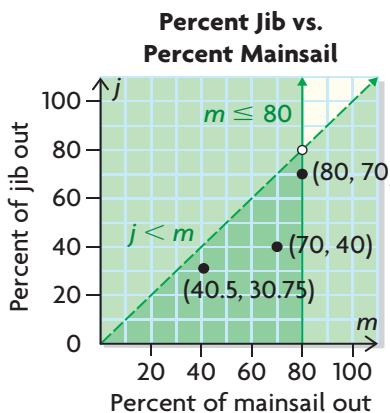
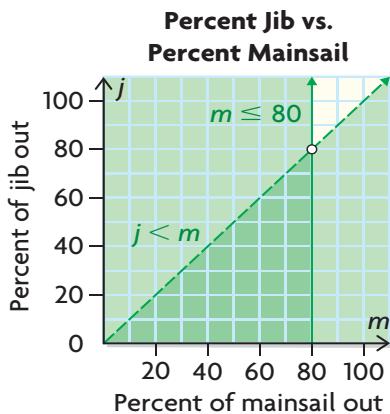
The inequalities describe the following information:

- No more than 80% of the mainsail can be out.
- Less jib than mainsail must be out.

I decided to use m as the independent variable.

I examined each inequality to determine its boundary:

- Since m is the independent variable, I knew the boundary for $m \leq 80$ would be a vertical line through $m = 80$.
- I knew the boundary of $j < m$ has a slope of 1 and passes through the point $(0, 0)$.



$$\{(m, j) \mid m \leq 80, m \geq 0, j \geq 0, m \in \mathbb{R}, j \in \mathbb{R}\}$$

$$\{(m, j) \mid j < m, m \geq 0, j \geq 0, m \in \mathbb{R}, j \in \mathbb{R}\}$$

Any point in the solution region represents an acceptable combination. For example,

- 80% of the mainsail and 70% of the jib can be out.
- 70% of the mainsail and 40% of the jib can be out.
- 40.5% of the mainsail and 30.75% of the jib can be out.

Your Turn

Suppose Jim wanted to have

- at least twice as much jib as mainsail out, and
- no more than 90% of the jib out.

How would the solution region change?

For $m \leq 80$, I drew a solid green vertical line through $m = 80$ and then shaded the half plane to its left green, since the inequality sign is \leq .

For $j < m$, I drew a green dashed line through $(0, 0)$ with a slope of 1 and I shaded the half plane below green since the inequality is $<$.

I drew an open dot where the dashed boundary intersects the solid boundary to show that point isn't part of the solution region.

The solution region for the system is a right triangle and consists of all the points in the overlapping region, including the solid boundary and the m -axis from 0 to 80.

I looked for several solutions in the solution region. I knew that I could choose points with decimal coordinates since the solution region is continuous.

In Summary

Key Ideas

- When graphing a system of linear inequalities, the boundaries of its solution region may or may not be included, depending on the types of linear inequalities (\geq , \leq , $<$, or $>$) in the system.
- Most systems of linear inequalities representing real-world situations are restricted to the first quadrant because the values of the variables in the system must be positive.

Need to Know

- Any point in the solution region for a system is a valid solution, but some solutions may make more sense than others depending on the context of the problem.
- You can validate a possible solution from the solution region by checking to see if it satisfies each linear inequality in the system.

For example, to validate if $(2, 2)$ is a solution to the system:

$$x + y \geq 1$$

$$2 > x - 2y$$

Validating $(2, 2)$ for $x + y \geq 1$:

LS	RS
$x + y$	1
$2 + 2$	
4	

$$4 \geq 1$$

valid

Validating $(2, 2)$ for $2 > x - 2y$:

LS	RS
$x - 2y$	
$2 - 2(2)$	
-2	

$$2 > -2$$

valid

- Use an open dot to show that an intersection point of a system's boundaries is excluded from the solution set. An intersection point is excluded when a dashed line intersects either a dashed or solid line.
- Use a solid dot to show that an intersection point of a system's boundaries is included in the solution set. This occurs when both boundary lines are solid.

CHECK Your Understanding

1. Graph the solution set for each system of inequalities. Determine a solution. Check its validity.

a) $x + y \geq 2$
 $x < 4$

b) $x + 2y < 6$
 $x < y$

c) $2x - 4 \geq y$
 $2y + 3x \leq 7$

2. a) Graph the solution set for this system of inequalities. Determine a solution. Check its validity. Describe the solution region.

$$x \leq 6$$

$$3y - x < 6$$

- b) Determine if each point is in the solution region.

- i) $(6, 4)$ ii) $(8, 2)$ iii) $(3, 2)$ iv) $(3, 3)$

- 3.** For each system of linear inequalities, explain whether the boundaries and their points of intersection are part of the solution region.
- $\{(x, y) \mid y \geq -2x, x \in \mathbb{R}, y \in \mathbb{R}\}$
 $\{(x, y) \mid -3 < x - y, x \in \mathbb{R}, y \in \mathbb{R}\}$
 - $\{(x, y) \mid x + y \leq -2, x \in \mathbb{I}, y \in \mathbb{I}\}$
 $\{(x, y) \mid 2y \geq x, x \in \mathbb{I}, y \in \mathbb{I}\}$
 - $\{(x, y) \mid x + 3y \geq 0, x \in \mathbb{I}, y \in \mathbb{I}\}$
 $\{(x, y) \mid x + y > 2, x \in \mathbb{I}, y \in \mathbb{I}\}$

PRACTISING

- 4.** Graph each system. Determine a solution for each.
- $\{(x, y) \mid x + y \leq 3, x \in \mathbb{R}, y \in \mathbb{R}\}$
 $\{(x, y) \mid y > 2, x \in \mathbb{R}, y \in \mathbb{R}\}$
 - $\{(x, y) \mid 2x + y > 0, x \in \mathbb{W}, y \in \mathbb{W}\}$
 $\{(x, y) \mid y > x, x \in \mathbb{W}, y \in \mathbb{W}\}$
 - $\{(x, y) \mid 3y - 2x \leq 6, x \in \mathbb{N}, y \in \mathbb{N}\}$
 $\{(x, y) \mid 2y - 3x \leq 6, x \in \mathbb{N}, y \in \mathbb{N}\}$
 - $\{(x, y) \mid y - x \geq 3, x \in \mathbb{R}, y \in \mathbb{R}\}$
 $\{(x, y) \mid y + 2 \leq x, x \in \mathbb{R}, y \in \mathbb{R}\}$
- 5.** For each system of linear inequalities,
- graph the solution set.
 - describe the solution region.
 - determine a solution. Check its validity.
 - $\{(x, y) \mid x + y > 5, x \in \mathbb{R}, y \in \mathbb{R}\}$
 $\{(x, y) \mid y \leq 4, x \in \mathbb{R}, y \in \mathbb{R}\}$
 - $\{(x, y) \mid x + y \geq 1, x \in \mathbb{N}, y \in \mathbb{N}\}$
 $\{(x, y) \mid 2 > x - 2y, x \in \mathbb{N}, y \in \mathbb{N}\}$
 - $\{(x, y) \mid 4x + 11y > 44, x \in \mathbb{W}, y \in \mathbb{W}\}$
 $\{(x, y) \mid 2x - 6 \leq 3y, x \in \mathbb{W}, y \in \mathbb{W}\}$
- 6.** The staff in a cafeteria are making two kinds of sandwiches: egg salad, and ham and cheese:
- A maximum of 450 sandwiches are needed.
 - Based on previous demand, there should be at least twice as many ham and cheese sandwiches as egg salad sandwiches.
- Define the variables and write a system of inequalities that models this situation.
 - Describe the restrictions on the variables in this situation.
 - Graph the system to determine the solution set.
 - Suggest two combinations of numbers of sandwiches that the cafeteria staff could make.



7. a) Graph the solution set for the following system of inequalities.

Determine a solution. Check its validity.

$$9x + 18y < 18$$

$$3x - 6y \leq 18$$

- b) Is each point below a possible solution to the system?

How do you know?

i) $(4, -1)$

ii) $(-2, 2)$

iii) $(-4, -2)$

iv) $(9, 1)$

v) $(-2.5, -1.5)$

vi) $(2, -2)$

8. Trish is setting up her social networking page:

- She wants to have no more than 500 friends on her new social networking page.
- She also wants to have at least three school friends for every rugby friend.
 - Define the variables and write a system of inequalities that models this situation.
 - Describe the restrictions on the domain and range of the variables.
 - Graph the solution set to determine two possible combinations of school friends and rugby friends she could have.

9. Graph the solution set for this system of linear inequalities to determine two valid solutions:

$$\{(x, y) \mid 3x + y \leq 2, x \in I, y \in I\}$$

$$\{(x, y) \mid 2y + 3x > 1, x \in I, y \in I\}$$

10. Spence, a disc jockey, is often hired to play weddings.

- His contract states that he will play no longer than 3 h, with no more than 12 songs each hour.
- He likes to play two or more songs for young listeners for every one song he plays for older listeners.

Determine three possible combinations of numbers of songs he could play.



Closing

11. Create a system of two linear inequalities for each situation.

- The boundaries of the solution region are included in the solution set, and all the points in the solution region are valid solutions.
- The boundaries of the solution region are not included in the solution set, and only whole-number points in the solution region are valid solutions.

Extending

12. Graph the solution set for the following system of inequalities.

Describe the solution region.

$$2y + 8 \leq x$$

$$-4 + y + x < 0$$

$$3x + y + 3 > 3(1 + x)$$

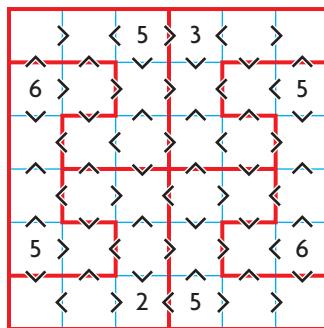
Applying Problem-Solving Strategies

Inequality Sudoku

Sudoku puzzles are seen in many books, newspapers, and magazines. However, inequality Sudoku puzzles, like the one shown below, are much less common.

The Puzzle

- A. Complete the puzzle by inserting the digits 1 through 6 into the cells so that the inequality signs are correct. Each digit must occur only once in each column or row, as in a conventional Sudoku puzzle. Also, the digits 1 to 6 must occur within each region outlined in red.



The Strategy

- B. Describe what you know about the rows and columns and the given numbers.
C. Describe how you used deductive reasoning to complete the puzzle.

Variation

- D. Start with a blank six-by-six grid. Fill in the digits 1 to 6 so that each digit occurs only once in each row and column. Place inequality signs between the digits. This completed puzzle is your answer key.
E. Prepare a blank copy of your six-by-six grid, with just the inequality signs and a few numbers.
F. Switch puzzles with a classmate, and solve each other's puzzles using deductive reasoning.
G. If you are unable to find a starting point, ask your classmate to place a 6 or a 1 in its correct place in the puzzle.