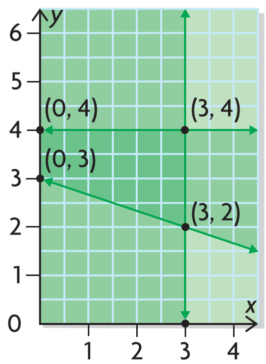
6.6 Optimization Problems III: Linear Programming

YOU WILL NEED

- graphing technology OR graph paper, ruler, and coloured pencils

EXPLORE...

- The following system of linear inequalities has been graphed below:



System of linear inequalities:
$y \geq 0$
$x \geq 0$
$y \leq 4$
$x \leq 3$
$3y \geq -x + 9$

a) For each objective function, what points in the feasible region represent the minimum and maximum values?
i) $T = 5x + y$
ii) $T = x + 5y$
b) What do you notice about the optimal points for the two objective functions? Why do you think this happened?

GOAL

Solve optimization problems.

LEARN ABOUT the Math

In Lesson 6.4, you created an optimization model for the problem below. In Lesson 6.5, you explored the model for optimal solutions.

A toy company manufactures two types of toy vehicles: racing cars and sport-utility vehicles.

- Because the supply of materials is limited, no more than 40 racing cars and 60 sport-utility vehicles can be made each day.
- However, the company can make 70 or more vehicles, in total, each day.
- It costs $8 to make a racing car and $12 to make a sport-utility vehicle.

There are many possible combinations of racing cars and sport-utility vehicles that could be made. The company wants to know what combinations will result in the minimum and maximum costs, and what those costs will be.

? Which combinations of the two types of vehicles will result in a minimum and a maximum production cost, and what will these costs be?

EXAMPLE 1 Solving an optimization problem by graphing an algebraic model

Ramona's Solution

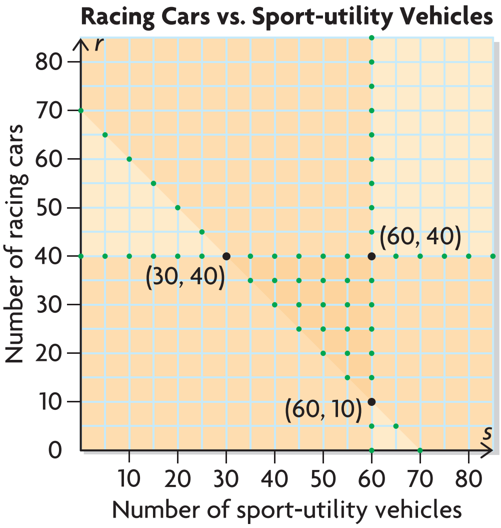
Let s represent sport-utility vehicles.
Let r represent racing cars.

I started by creating an algebraic model to represent the situation.

Restrictions on the variables:
$r \in W, s \in W$

Constraints:
$r \geq 0$
$s \geq 0$
$r \leq 40$
$s \leq 60$
$r + s \geq 70$

Let C represent the total production cost.
Objective function to optimize, at $12 per sport-utility vehicle and $8 per car:
$C = 12s + 8r$



I graphed the system of inequalities to determine the points at the vertices of the feasible region.

I knew that two of these points represent the solutions that will optimize the objective function.

I conjectured that

- the minimum solution is (30, 40) because this represents the lowest total number of vehicles (70) and there are fewer of the more expensive sport-utility vehicles than racing cars (30 versus 40).
- the maximum solution is (60, 40) because this represents the highest total number of vehicles (100) and there is a greater number of the more expensive sport-utility vehicles than racing cars (60 vs. 40).

Optimize:
$C = 12s + 8r$

If (s, r) is (60, 10), $C = 12(60) + 8(10)$ $C = 720 + 80$ $C = \$800$	If (s, r) is (60, 40), $C = 12(60) + 8(40)$ $C = 720 + 320$ $C = \$1040$	If (s, r) is (30, 40), $C = 12(30) + 8(40)$ $C = 360 + 320$ $C = \$680$

To check my conjecture, I evaluated the objective function using the coordinates of each vertex of the feasible region. This enabled me to compare the production cost for each solution.

The company can minimize the production cost to \$680 by making 30 sport-utility vehicles and 40 racing cars and maximize costs to \$1040 by making 60 sport-utility vehicles and 40 racing cars.

My conjecture was correct.

Verify minimum (30, 40):

(s, r) is (30, 40): $r \leq 40$ LS: 40 | RS: 40 $40 \leq 40$	(s, r) is (30, 40): $s \leq 60$ LS: 30 | RS: 60 $30 \leq 60$	(s, r) is (30, 40): $r + s \geq 70$ LS: 30 + 40 | RS: 70 $70 \geq 70$

Verify minimum (60, 40):

(s, r) is (60, 40): $r \leq 40$ LS: 40 | RS: 40 $40 \leq 40$	(s, r) is (60, 40): $s \leq 60$ LS: 60 | RS: 60 $60 \leq 60$	(s, r) is (60, 40): $r + s \geq 70$ LS: 60 + 40 | RS: 70 $100 \geq 70$

I verified each optimal solution to make sure it satisfied every constraint in the system. Each inequality statement was valid for both optimal solutions.

Reflecting

A. How would the solution change if the cost to make a racing car was \$12 and the cost to make a sport-utility vehicle was \$8? How could you have predicted this?

B. How would the solution change if the cost was \$10 for each vehicle? How could you have predicted that?

C. Summarize the steps you can follow when using **linear programming** to solve an optimization problem.

linear programming

A mathematical technique used to determine which solutions in the feasible region result in the optimal solutions of the objective function.

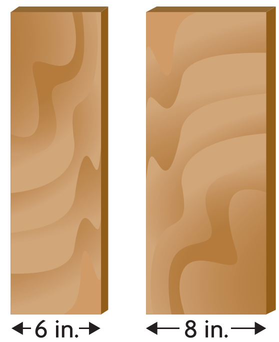
APPLY the Math

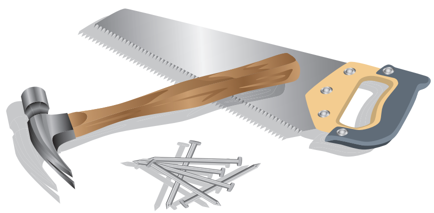
EXAMPLE 2 Creating a model for an optimization problem, and solving the problem

L&G Construction is competing for a contract to build a fence.

- The fence will be no longer than 50 yd and will consist of narrow boards that are 6 in. wide and wide boards that are 8 in. wide.
- There must be no fewer than 100 wide boards and no more than 80 narrow boards.
- The narrow boards cost \$3.56 each, and the wide boards cost \$4.36 each.

Determine the maximum and minimum costs for the lumber to build the fence.





Aisla's Solution

Let n represent the number of narrow boards.
Let w represent the number of wide boards.
Let C represent the total cost of the lumber.

I began by defining the variables in this situation.

Restrictions:
$n \in \text{W}$ and $w \in \text{W}$

Since both n and w represent the number of boards needed, they must be whole numbers.

No fewer than 100 wide boards:
$w \geq 100$
No more than 80 narrow boards:
$n \leq 80$
The fence is no longer than 50 yd and is made of 6 in. narrow and 8 in. wide boards:
$6n + 8w \leq 50(36)$
$6n + 8w \leq 1800$

Next, I created the constraints in the problem. I represented these constraints with linear inequalities.

I converted yards to inches using the rate 36 in./1 yd, since the coefficients of the other terms are in inches.

Optimization Model

Constraints:

$n \geq 0$

$w \geq 0$

$w \geq 100$

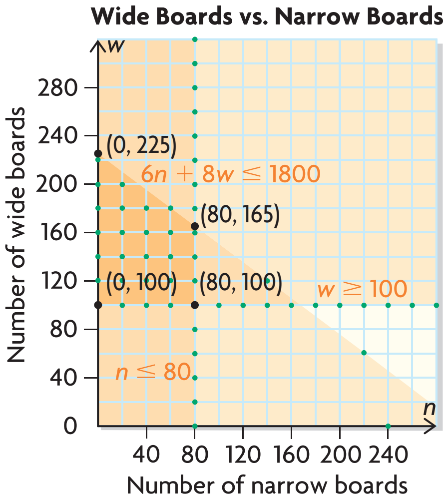
$n \leq 80$

$6n + 8w \leq 1800$

Objective function to optimize at $3.56 per narrow board and $4.36 per wide board:

$C = 3.56n + 4.36w$

I created the objective function that represents the relationship between the cost of the two widths of boards and the total cost of the lumber.



I graphed in the first quadrant because the domain and range are restricted to whole numbers. I decided to make the the number of narrow boards (n) the independent variable.

I identified the coordinates of the vertices of the feasible region.

I made these conjectures:

- Point (80, 165) will result in the maximum cost since it is farthest from both axes, which means it has large coordinate values.
- Point (0, 100) will result in the minimum cost, since one coordinate is 0 and the other coordinate is only 100.

$C = 3.56n + 4.36w$

If (n, w) is (0, 100), $C = 3.56(0) + 4.36(100)$ $C = \$436.00$	If (n, w) is (0, 225), $C = 3.56(0) + 4.36(225)$ $C = \$981.00$
If (n, w) is (80, 100), $C = 3.56(80) + 4.36(100)$ $C = \$720.80$	If (n, w) is (80, 165), $C = 3.56(80) + 4.36(165)$ $C = \$1004.20$

I substituted the values of n and w for all four vertices into the objective function to compare the cost for these solutions.

(0, 100), or no narrow boards and 100 wide boards, cost the minimum amount: $436.

(80, 165), or 80 narrow boards and 165 wide boards, cost the maximum amount: $1004.20.

My conjectures were correct.

Verify minimum (0, 100):

(n, w) is (0, 100): $n \le 80$ LS: 0	RS: 80 $0 \le 80$	(n, w) is (0, 100): $w \ge 100$ LS: 100	RS: 100 $100 \ge 100$	(n, w) is (0, 100): $6n + 8w \le 1800$ LS: $6(0) + 8(100)$, 800	RS: 1800 $800 \le 1800$

Verify maximum (80, 165):

(n, w) is (80, 165): $n \le 80$ LS: 80	RS: 80 $80 \le 80$	(n, w) is (80, 165): $w \ge 100$ LS: 165	RS: 100 $165 \ge 100$	(n, w) is (80, 165): $6n + 8w \le 1800$ LS: $6(80) + 8(165)$, 1800	RS: 1800 $1800 \le 1800$

I verified both solutions to make sure they satisfied every constraint in the system. Each inequality statement was valid for both optimal solutions.

Your Turn

How would the feasible region change if 80 or more narrow boards had to be used?

In Summary

Key Idea

- To solve an optimization problem using linear programming, begin by creating algebraic and graphical models of the problem (as shown in Lesson 6.4). Then use the objective function to determine which vertex of the feasible region results in the optimal solution.

Need to Know

- The solution to an optimization problem is usually found at one of the vertices of the feasible region.
- To determine the optimal solution to an optimization problem using linear programming, follow these steps:

  Step 1. Create an algebraic model that includes:
  - a defining statement of the variables used in your model
  - the restrictions on the variables
  - a system of linear inequalities that describes the constraints
  - an objective function that shows how the variables are related to the quantity to be optimized

  Step 2. Graph the system of inequalities to determine the coordinates of the vertices of its feasible region.

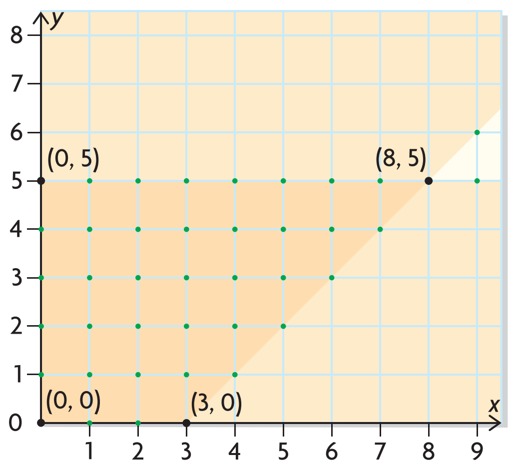
  Step 3. Evaluate the objective function by substituting the values of the coordinates of each vertex.

  Step 4. Compare the results and choose the desired solution.

  Step 5. Verify that the solution(s) satisfies the constraints of the problem situation.

CHECK Your Understanding

1. Determine the optimal solutions for the system of linear inequalities graphed below, using the objective function $G = 2x + 5y$.



2. The following model represents an optimization problem. Determine the maximum solution.

Optimization Model

Restrictions:
$x \in \mathrm{R}$ and $y \in \mathrm{R}$

Constraints:
$x \geq 0$
$y \leq 0$
$3y \geq -2x + 3$
$y \geq 2x - 7$

Objective function:
$D = -5x + 3y$

3. Three teams are travelling to a basketball tournament in cars and minivans.

- Each team has no more than 2 coaches and 14 athletes.
- Each car can take 4 team members, and each minivan can take 6 team members.
- No more than 4 minivans and 12 cars are available.

The school wants to know the combination of cars and minivans that will require the maximum number of vehicles. Create and verify a model to represent this situation.

a) Use the optimization model to determine the combination of cars and minivans that will use the maximum number of vehicles.

b) How many team members can travel in the maximum number of vehicles?

Optimization Model

Let V represent the total number of vehicles.
Let c represent the number of cars.
Let m represent the number of minivans.

Restrictions:
$c \in \mathrm{W}, m \in \mathrm{W}$

Constraints:
$c \geq 0$
$m \geq 0$
$4c + 6m \leq 48$
$c \leq 12$
$m \leq 4$

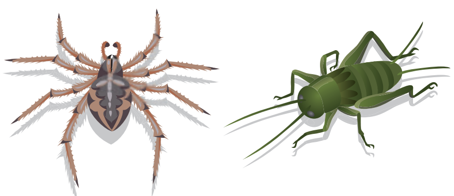
Objective function to maximize:
$V = c + m$

4. Ed found spiders and crickets in his storage room.

- There were 20 or fewer spiders and 20 or more crickets.
- There were 45 or fewer crickets and spiders, in total. Spiders have 8 legs, and crickets have 6 legs.

a) What combination of spiders and crickets would have the greatest number of legs?

b) What combination would have the least number of legs?



PRACTISING

5. The following model represents an optimization problem. Determine the maximum solution.

Optimization Model
Restrictions:
$x \in \mathrm{W}, y \in \mathrm{W}$

Constraints:
$x \geq 0$
$y \geq 0$
$x + y \leq 5$
$2x + y \geq 5$

Objective function:
$K = -x + 2y$

6. The following model represents an optimization problem. Determine the minimum solution.

Optimization Model
Restrictions:
$x \in \mathrm{W}, y \in \mathrm{W}$

Constraints:
$x \geq 0$
$y \geq 0$
$3x + y \geq 15$
$x \leq 10$
$x \leq 9$

Objective function:
$P = 5x + 3y$

7. The following model represents an optimization problem. Determine the maximum solution.

Optimization Model
Restrictions:
$m \in \mathrm{R}, s \in \mathrm{R}$

Constraints:
$m \geq 0$
$s \geq 0$
$3m + 4s \leq 24$
$m + s \geq 4$

Objective function:
$T = 1.5m + 4.2s$

8. A refinery produces oil and gas.
- At least 2 L of gasoline is produced for each litre of heating oil.
- The refinery can produce up to 9 million litres of heating oil and 6 million litres of gasoline each day.
- Gasoline is projected to sell for $1.10 per litre. Heating oil is projected to sell for $1.75 per litre.

The company needs to determine the daily combination of gas and heating oil that must be produced to maximize revenue. Create a model to determine this combination. What would the revenue be?

Optimization Model

Let g represent the number of litres of gasoline.
Let h represent the number of litres of heating oil.
Let R represent the total revenue from sales.

Restrictions:
$g \in \mathrm{R}, h \in \mathrm{R}$

Constraints:
$g \geq 0$
$h \geq 0$
$g \geq 2h$
$g \leq 6\,000\,000$
$h \leq 9\,000\,000$

Objective function to maximize:
$R = 1.10g + 1.75h$

9. Northwest Trail Mix Limited (NTML) is preparing 1 kg bags of nuts to sell.
- NTML decides to make and sell no fewer than 3000 bags of walnuts, and no more than 5000 bags of almonds.
- The marketing department has predicted sales of no fewer than 6000 bags altogether.
- NTML wants to minimize costs.

The cost per kilogram is shown in the chart.

Type of Nut	Cost
almonds 	$11.19/kg
walnuts 	$13.10/kg

a) Write a system of linear inequalities to describe these constraints:
  i) the number of bags of almonds
  ii) the number of bags of walnuts
  iii) the total number of bags to be sold

b) Describe the restrictions on the domain and range of the variables.
c) Graph the system of linear inequalities.
d) Describe the feasible region.
e) Write the objective function to represent the quantity to be minimized.
f) Determine the minimum cost for NTML.

10. Choose two optimization models that were developed in the Practising questions in Lesson 6.4. Solve each model.

11. On a flight between Winnipeg and Vancouver, there are business class and economy seats.

- At capacity, the airplane can hold no more than 145 passengers.
- No fewer than 130 economy seats are sold, and no more than 8 business class seats are sold.
- The airline charges \$615 for business class seats and \$245 for economy seats.

What combination of business class and economy seats will result in the maximum revenue? What will this maximum revenue be?

12. A school is organizing a track and field meet.

- There will be no more than 250 events and no fewer than 100 events to be scheduled.
- The organizers allow 15 min for each track event and 45 min for each field event.
- They are considering different combinations of track and field events.

What are the least and greatest amounts of time they should allow?

13. Sophie has two summer jobs.

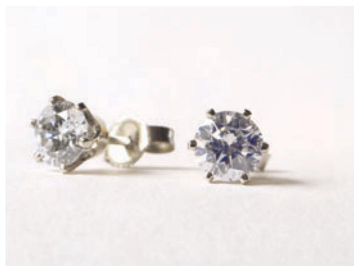
- She works no more than a total of 32 h a week. Both jobs allow her to have flexible hours but in whole hours only.
- At one job, Sophie works no less than 12 h and earns \$8.75/h.
- At the other job, Sophie works no more than 24 h and earns \$9.00/h.

What combination of numbers of hours will allow her to maximize her earnings? What can she expect to earn?

14. A jewellery store sells diamond earrings: small earrings (no more than 1 carat of diamonds) and large earrings (more than 1 carat of diamonds).

- They sell at least four pairs of small earrings for every pair of large earrings.
- They also sell no more than 120 pairs of earrings, in total, per month.
- The small earrings sell for about \$800 a pair, and the large earrings sell for about \$1500 a pair.

What combination of the two categories of earrings should they try to sell to maximize their revenue? What amount of sales can they expect?



15. A hardware store sells both asphalt shingles and cedar shakes as roofing materials.
    - The store stocks at least eight times as many bundles of asphalt shingles as cedar shakes.
    - They also stock at least 200 bundles of cedar shakes.
    - The storeroom has space for no more than 2000 bundles altogether.
    - Each bundle of cedar shakes takes up 1.5 cubic feet, and each bundle of asphalt shingles takes up 1 cubic foot.

    What combination of asphalt shingles and cedar shakes should the store stock to minimize its storage needs and still keep enough roofing materials on hand for sales?

Closing

16. In Lesson 6.4, question 8, you developed a list of questions that could help you create a model for an optimization problem. Develop another list of questions that could help you use this model to solve the problem.

Extending

17. Choose one of these contexts, or create your own context, and write an optimization problem based on this context. Trade problems with a partner, and solve each other's problem.
    - using a rental van or a moving company to move
    - building a library with hardcover and paperback books
    - watching movies by going to a theatre or by renting them